An Introduction to Stochastic Processes in Continuous Time:
the non-Jip-and-Janneke-language approach

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adaptation of the text by
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to be used at your own expense

May 6, 2018
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1.1 Introduction

Loosely speaking, a stochastic process is a phenomenon that can be thought of as evolving in time in a random manner. Common examples are the location of a particle in a physical system, the price of stock in a financial market, interest rates, mobile phone networks, internet traffic, etc.

A basic example is the erratic movement of pollen grains suspended in water, so-called Brownian motion. This motion was named after the English botanist R. Brown, who first observed it in 1827. The movement of pollen grain is thought to be due to the impacts of water molecules that surround it. Einstein was the first to develop a model for studying the erratic movement of pollen grains in an article in 1926. We will give a sketch of how this model was derived. It is more heuristically than mathematically sound.

The basic assumptions for this model (in dimension 1) are the following:

1) the motion is continuous.

Moreover, in a time-interval \([t, t + \tau]\), \(\tau\) small,

2) particle movements in two non-overlapping time intervals of length \(\tau\) are mutually independent;

3) the relative proportion of particles experiencing a displacement of size between \(\delta\) and \(\delta + d\delta\) is approximately \(\phi_r(\delta)\) with
   - the probability of some displacement is 1: \(\int_{-\infty}^{\infty} \phi_r(\delta)d\delta = 1\);
   - the average displacement is 0: \(\int_{-\infty}^{\infty} \delta \phi_r(\delta)d\delta = 0\);
   - the variation in displacement is linear in the length of the time interval: \(\int_{-\infty}^{\infty} \delta^2 \phi_r(\delta)d\delta = D\tau\), where \(D \geq 0\) is called the diffusion coefficient.

Denote by \(f(x, t)\) the density of particles at position \(x\), at time \(t\). Under differentiability assumptions, we get by a first order Taylor expansion that

\[
f(x, t + \tau) \approx f(x, t) + \tau \frac{\partial f}{\partial t}(x, t).
\]
On the other hand, by a second order expansion

\[ f(x, t + \tau) = \int_{-\infty}^{\infty} f(x - \delta, t) \phi_\tau(\delta) d\delta \]

\[ \approx \int_{-\infty}^{\infty} [f(x, t) - \delta \frac{\partial f}{\partial x}(x, t)] \phi_\tau(\delta) d\delta + \frac{1}{2} \delta^2 \frac{\partial^2 f}{\partial x^2}(x, t) \phi_\tau(\delta) d\delta \]

\[ \approx f(x, t) + \frac{1}{2} D \frac{\partial^2 f}{\partial x^2}(x, t). \]

Equating gives rise to the heat equation in one dimension:

\[ \frac{\partial f}{\partial t} = \frac{1}{2} D \frac{\partial^2 f}{\partial x^2}, \]

which has the solution

\[ f(x, t) = \frac{\#\text{particles}}{\sqrt{4\pi Dt}} e^{-x^2/4Dt}. \]

So \( f(x, t) \) is the density of a \( \mathcal{N}(0, 4Dt) \)-distributed random variable multiplied by the number of particles.

**Side remark.** In section 1.5 we will see that under these assumptions paths of pollen grain through liquid are non-differentiable. However, from physics we know that the velocity of a particle is the derivative (to time) of its location. Hence pollen grain paths must be differentiable. We have a conflict between the properties of the physical model and the mathematical model. What is wrong with the assumptions? Already in 1926 editor R. Füth doubted the validity of the independence assumption (2). Recent investigation seems to have confirmed this doubt.

Brownian motion will be one of our objects of study during this course. We will now turn to a mathematical definition.

**Definition 1.1.1** Let \( T \) be a set and \((E, \mathcal{E})\) a measurable space. A **stochastic process** indexed by \( T \), with values in \((E, \mathcal{E})\), is a collection \( X = (X_t)_{t \in T} \) of measurable maps from a (joint) probability space \((\Omega, \mathcal{F}, P)\) to \((E, \mathcal{E})\). \( X \) is called a **random variable** again for simplicity. The space \((E, \mathcal{E})\) is called the state space of the process.

The index \( t \) is a time parameter, and we view the index set \( T \) as the set of all observation instants of the process. In these notes we will usually have \( T = \mathbb{Z}_+ = \{0, 1, \ldots\} \) or \( T = \mathbb{R}_+ = [0, \infty) \) (or \( T \) is a sub-interval of one these sets). In the former case, we say that time is **discrete**, in the latter that time is **continuous**. Clearly a discrete-time process can always be viewed as a continuous-time process that is constant on time-intervals \([n, n+1)\).

The state space \((E, \mathcal{E})\) will generally be a Euclidian space \( \mathbb{R}^d \), endowed with its Borel \( \sigma \)-algebra \( \mathcal{B}(\mathbb{R}^d) \). If \( E \) is the state space of the process, we call the process \( E \)-valued.

For every fixed observation instant \( t \in T \), the stochastic process \( X \) gives us an \( E \)-valued random variable \( X_t \) on \((\Omega, \mathcal{F}, P)\). We can also fix \( \omega \in \Omega \) and consider the map \( t \to X_t(\omega) \) on \( T \). These maps are called the **trajectories** or **sample paths** of the process. The sample paths are functions from \( T \) to \( E \) and so they are elements of the function space \( E^T \). Hence, we can view the process \( X \) as an \( E^T \)-valued random variable.
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Quite often, the sample paths belong to a nice subset of this space, e.g. the continuous or right-continuous functions, alternatively called the path space. For instance, a discrete-time process viewed as the continuous-time process described earlier, can be extended as a process with right-continuous sample paths.

Clearly we need to put an appropriate $\sigma$-algebra on the path space $E^T$. For consistency purposes, it is convenient that the marginal distribution of $X_t$ be a probability measure on the path space. This is achieved by ensuring that the projection $x \to x_t$, where $t \in T$, is measurable. The $\sigma$-algebra $E^T$, described in BN §2, is the minimal $\sigma$-algebra with this property.

We will next introduce the formal requirements for the stochastic processes that are called Brownian motion and Poisson process respectively. First, we introduce processes with independent increments.

**Definition 1.1.2** Let $E$ be a separable Banach space, and $E$ the Borel-$\sigma$-algebra of subsets of $E$. Let $T = [0, \tau] \subset \mathbb{R}^+$. Let $X = \{X_t\}_{t \in T}$ be an $(E, \mathcal{E})$-valued stochastic process, defined on an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

i) $X$ is called a process with independent increments, if $\sigma(X_t - X_s)$ and $\sigma(X_u, u \leq s)$, are independent for all $s \leq t \leq \tau$.

ii) $X$ is called a process with stationary, independent increments, if, in addition, $X_t - X_s \overset{d}{=} X_{t-s} - X_0$, for $s \leq t \leq \tau$.

The mathematical model of the physical Brownian motion is a stochastic process that is defined as follows.

**Definition 1.1.3** The stochastic process $W = (W_t)_{t \geq 0}$ is called a (standard) Brownian motion or Wiener process, if

i) $W_0 = 0$, a.s.;

ii) $W$ is a stochastic process with stationary, independent increments;

iii) $W_t - W_s \overset{d}{=} \mathcal{N}(0, t-s)$;

iv) almost all sample paths are continuous.

In these notes we will abbreviate ‘Brownian motion’ as BM. Property (i) tells that standard BM starts at 0. A stochastic process with property (iv) is called an a.s. continuous process. Similarly, a stochastic process is said to be a.s. right-continuous if almost all of its sample paths are right-continuous functions. If the addition ‘a.s.’ is omitted, then the corresponding property is meant to hold everywhere. Finally, the acronym cadlag (continu à droite, limites à gauche) is used for processes with right-continuous sample paths having finite left-hand limits at every time instant.

Simultaneously with Brownian motion we will discuss another fundamental process: the Poisson process.

**Definition 1.1.4** A real-valued stochastic process $N = (N_t)_{t \geq 0}$ is called a Poisson process if
Proof. We will show that this is a Poisson process. First note that

i) \( N \) is a counting process, i.o.w.
   a) \( N_t \) takes only values in \((\mathbb{Z}_+, 2^{\mathbb{Z}_+})\), \( t \geq 0 \);
   b) \( t \mapsto N_t \) is increasing, i.o.w. \( N_s \leq N_t, t \geq s \).

ii) \( N_0 = 0 \), a.s.;

iii) \( N \) is a stochastic process with stationary, independent increments.

Note: so far we do not know yet whether a BM process and a Poisson process exist at all! The Poisson process can be constructed quite easily and we will do so first before delving into more complex issues.

Construction of the Poisson process The construction of a Poisson process is simpler than the construction of Brownian motion. It is illustrative to do this first.

Let a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) be given. We construct a sequence of i.i.d. \((\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))\)-measurable random variables \( X_n, n = 1, \ldots \), on this space, such that \( X_n \overset{d}{=} \exp(\lambda) \). This means that

\[
\mathbb{P}(X_n > t) = e^{-\lambda t}, \quad t \geq 0.
\]

Put \( S_0 = 0 \), and \( S_n = \sum_{i=1}^{n} X_i \). Clearly \( S_n, n = 0, \ldots \) are in increasing sequence of random variables. Since \( X_n \) are all \( \mathcal{F}/\mathcal{B}(\mathbb{R}_+) \)-measurable, so are \( S_n \). Next define

\[
N_t = \max\{n \mid S_n \leq t\}.
\]

We will show that this is a Poisson process. First note that \( N_t \) can be described alternatively as

\[
N_t = \sum_{n=1}^{\infty} 1_{\{S_n \leq t\}}.
\]

\( N_t \) may be infinite, but we will show that it is finite with probability 1 for all \( t \). Moreover, no two points \( S_n \) and \( S_{n+1} \) are equal. Denote by \( \mathcal{E} \) the \( \sigma \)-algebra generated by the one-point sets of \( \mathbb{Z}_+ \).

Lemma 1.1.5 \( N_t \) is \( \mathcal{F}/2^{\mathbb{Z}_+} \)-measurable. There exists a set \( \Omega^* \in \mathcal{F} \) with \( \mathbb{P}\{\Omega^*\} = 1 \), such that \( N_t(\omega) < \infty \) for all \( t \geq 0, \omega \in \Omega^* \), and \( S_n(\omega) < S_{n+1}(\omega), n = 0, \ldots \).

Proof. From the law of large numbers we find a set \( \Omega' \), \( \mathbb{P}\{\Omega'\} = 1 \), such that \( N_t(\omega) < \infty \) for all \( t \geq 0, \omega \in \Omega' \). It easily follows that there exists a subset \( \Omega^* \subset \Omega' \), \( \mathbb{P}\{\Omega^*\} = 1 \), meeting the requirements of the lemma. Measurability follows from the fact that \( 1_{\{S_n \leq t\}} \) is measurable. Hence a finite sum of these terms is measurable. The infinite sum is then measurable as well, being the monotone limit of measurable functions.

QED

Since \( \Omega^* \in \mathcal{F} \), we may restrict to this smaller space without further ado. Denote the restricted probability space again by \((\Omega, \mathcal{F}, \mathbb{P})\).

Theorem 1.1.6 For the constructed process \( N \) on \((\Omega, \mathcal{F}, \mathbb{P})\) the following hold.

i) \( N \) is a \((\mathbb{Z}_+, \mathcal{E})\)-measurable stochastic process that has properties (i,...,iv) from Definition 1.1.4. Moreover, \( N_t \) is \( \mathcal{F}/\mathcal{E} \)-measurable, it has a Poisson distribution with parameter \( \lambda t \) and \( S_n \) has a Gamma distribution with parameters \( (n, \lambda) \). In particular \( \mathbb{E} N_t = \lambda t \), and \( \mathbb{E} N_t^2 = \lambda t + (\lambda t)^2 \).
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ii) All paths of $N$ are cadlag.

Proof. The second statement is true by construction, as well as are properties (i,ii). The fact that $N_t$ has a Poisson distribution with parameter $\lambda t$, and that $S_n$ has $\Gamma(n,\lambda)$ distribution is standard.

We will prove property (iv). It suffices to show for $t \geq s$ that $N_t - N_s$ has a Poisson $(\lambda(t-s))$ distribution. Clearly

$$P\{N_t - N_s = j\} = \sum_{i \geq 0} P\{N_s = i, N_t - N_s = j\} = \sum_{i \geq 0} P\{S_i \leq s, S_{i+1} > s, S_{i+j} \leq t, S_{i+j+1} > t\}. \tag{1.1.1}$$

First let $i, j > 1$. Recall the density $f_{n,\lambda}$ of the $\Gamma(n,\lambda)$ distribution:

$$f_{n,\lambda}(x) = \frac{\lambda^n x^{n-1} e^{-\lambda x}}{\Gamma(n)}, \quad n \neq 1$$

where $\Gamma(n) = (n-1)!$. Then, with a change of variable $u = s_2 - (s - s_1)$ in the third equation,

$$P\{N_t - N_s = j, N_s = i\} = P\{S_i \leq s, S_{i+1} > s, S_{i+j} \leq t, S_{i+j+1} > t\}
= \int_0^s \int_{s-s_1}^{t-s_1} \int_{s-s_2-s_1}^{t-s_1} e^{-\lambda(t-s_3-s_2-s_1)} f_{j-1,\lambda}(s_3) ds_3 \lambda e^{-\lambda s_2} ds_2 f_{i,\lambda}(s_1) ds_1
= \int_0^s \int_0^{t-s} \int_0^{t-s-u} e^{-\lambda(t-s-u-s_3)} f_{j-1,\lambda}(s_3) ds_3 \lambda e^{-\lambda u} du \cdot e^{-\lambda(s-s_1)} f_{i,\lambda}(s_1) ds_1
= P\{S_j \leq t - s, S_{j+1} > t - s\} \cdot P\{S_i \leq s, S_{i+1} > s\}
= P\{N_{t-s} = j\} P\{N_s = i\}. \tag{1.1.2}$$

For $i = 0, j = 1$, we get the same conclusion. (1.1.1) then implies that $P\{N_t - N_s = j\} = P\{N_{t-s} = j\}$, for $j > 0$. By summing over $j > 0$ and substracting from 1, we get the relation for $j = 0$ and so we have proved property (iv).

Finally, we will prove property (iii). Let us first consider $\sigma(N_u, u \leq s)$. This is the smallest $\sigma$-algebra that makes all maps $\omega \mapsto N_u(\omega)$, $u \leq s$, measurable. Section 2 of BN studies its structure. It follows that (see Exercise 1.1) the collection $\mathcal{I}$, with

$$\mathcal{I} = \left\{ A \in \mathcal{F} \mid \exists n \in \mathbb{Z}_+, t_0 \leq t_1 < t_2 < \cdots < t_n, t_l \in [0, s], i_l \in \mathbb{Z}_+, l = 0, \ldots, n, \right.$$ such that $A = \{N_{t_l} = i_l, l = 0, \ldots, n\} \right\}$$

a $\pi$-system for this $\sigma$-algebra.

To show independence property (iii), it therefore suffices show for each $n$, for each sequence $0 \leq t_0 < \cdots < t_n$, and $i_0, \ldots, i_n$, that

$$P\{N_{t_l} = i_l, l = 0, \ldots, n, N_t - N_s = i\} = P\{N_{t_l} = i_l, l = 0, \ldots, n\} \cdot P\{N_t - N_s = i\}.$$

This is analogous to the proof of (1.1.2). QED
A final observation. We have constructed a mapping \( N : \Omega \to \Omega' \subset \mathbb{Z}_+^{[0,\infty)} \), with \( \mathbb{Z}_+^{[0,\infty)} \) the space of all integer-valued functions. The space \( \Omega' \) consists of all integer valued paths \( \omega' \), that are right-continuous and non-decreasing, and have the property that \( \omega'_t \leq \lim_{s \to t} \omega'_s + 1 \).

It is desirable to consider \( \Omega' \) as the underlying space. One can then construct a Poisson process directly on this space, by taking the identity map. This will be called the canonical process. The \( \sigma \)-algebra to consider, is then the minimal \( \sigma \)-algebra \( \mathcal{F}' \) that makes all maps \( \omega' \mapsto \omega'_t \) measurable, \( t \geq 0 \). It is precisely \( \mathcal{F}' = \mathcal{E}^{[0,\infty)} \cap \Omega' \).

Then \( \omega \mapsto N(\omega) \) is measurable as a map \( \Omega \to \Omega' \). On \( (\Omega', \mathcal{F}') \) we now put the induced probability measure \( P' \) by \( P'\{A\} = P(\omega \mid N(\omega) \in A) \).

In order to construct BM, we will next discuss a procedure to construct a stochastic process, with given marginal distributions.

### 1.2 Finite-dimensional distributions

In this section we will recall Kolmogorov’s theorem on the existence of stochastic processes with prescribed finite-dimensional distributions. We will use the version that is based on the fact that \( T \) is ordered. It allows to prove the existence of a process with properties (i,ii,iii) of Definition 1.1.3.

**Definition 1.2.1** Let \( X = (X_t)_{t \in T} : (\Omega, \mathcal{F}, P) \to (E^T, \mathcal{E}) \) be an \((E, \mathcal{E})\)-valued stochastic process. The distributions of the finite-dimensional vectors of the form \((X_{t_1}, X_{t_2}, \ldots, X_{t_n})\), \( t_1 < t_2 < \cdots < t_n \), are called the finite-dimensional distributions (fdd’s) of the process.

Recall that the \( n \)-rectangles form a \( \pi \)-system for \( \mathcal{E}^T \). More precisely, put

\[
\mathcal{I} = \left\{ A \in \mathcal{E}^T \left| \begin{array}{l}
\text{there exist } n \geq 1, t_1, \ldots, t_n \in T, \text{ with } t_1 < t_2 < \cdots < t_n, \text{ and } A_1, \ldots, A_n \in \mathcal{E}, \\
\text{such that } A = \{ x \in E^T \mid (x_{t_1}, \ldots, x_{t_n}) \in A_1 \times \cdots \times A_n \} \right. \right\}
\]

Then \( \mathcal{I} \) is a \( \pi \)-system, with \( \sigma(\mathcal{I}) = \mathcal{E}^T \). Thus, \( X^{-1}(\mathcal{I}) \subset \mathcal{F} \) is a \( \pi \)-system generating \( \sigma(X) \), and therefore the fdd’s define the probability distribution of \( X \).

It is easily verified that the fdd’s of a stochastic process form a consistent system of measures in the sense of the following definition.

**Definition 1.2.2** Let \( T \subset \mathbb{R} \) and let \((E, \mathcal{E})\) be a measurable space. For all \( n \in \mathbb{Z}_+ \) and all \( t_1 < \cdots < t_n, t_i \in T, i = 1, \ldots, n \), let \( \mu_{t_1, \ldots, t_n} \) be a probability measure on \((E^n, \mathcal{E}^n)\). This collection of measures is called consistent if it has the property that

\[
\mu_{t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n}(A_1 \times \cdots \times A_{i-1} \times A_{i+1} \times \cdots \times A_n) = \mu_{t_1, \ldots, t_n}(A_1 \times \cdots \times A_{i-1} \times E \times A_{i+1} \times \cdots \times A_n),
\]

for all \( A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n \in \mathcal{E} \).

The Kolmogorov consistency theorem states that, conversely, under mild regularity conditions, every consistent family of measures is in fact the family of fdd’s of some stochastic process.

Some assumptions are needed on the state space \((E, \mathcal{E})\). We will assume that \( E \) is a Polish space. This is a topological space, on which we can define a metric that consistent with the topology, and which makes the space complete and separable. As \( \mathcal{E} \) we take the Borel-\( \sigma \)-algebra, i.e. the \( \sigma \)-algebra generated by the open sets. Clearly, the Euclidian spaces \((\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))\) fit in this framework.
1.2. FINITE-DIMENSIONAL DISTRIBUTIONS

**Theorem 1.2.3 (Kolmogorov’s consistency theorem)** Suppose that $E$ is a Polish space and $\mathcal{E}$ its Borel-$\sigma$-algebra. Let $T \subset \mathbb{R}$ and for all $n \in \mathbb{Z}_+$, $t_1 < \ldots < t_n \in T$, let $\mu_{t_1,\ldots,t_n}$ be a probability measure on $(E^n, \mathcal{E}^n)$. If the measures $\mu_{t_1,\ldots,t_n}$ form a consistent system, then there exists a probability measure $\mathbb{P}$ on $E^T$, such that the canonical (or co-ordinate variable) process $(X_t)_t$ on $(\Omega = E^T, \mathcal{F} = \mathcal{E}^T, \mathbb{P})$, defined by $X(\omega) = \omega$, $X_t(\omega) = \omega_t$, has fdd’s $\mu_{t_1,\ldots,t_n}$.

The proof can for instance be found in Billingsley (1995). Before discussing this theorem, we will discuss its implications for the existence of $\text{BM}$.

**Corollary 1.2.4** There exists a probability measure $\mathbb{P}$ on the space $(\Omega = \mathbb{R}^{[0,\infty)}, \mathcal{F} = \mathcal{B}(\mathbb{R})^{[0,\infty)})$, such that the co-ordinate process $W = (W_t)_{t \geq 0}$ on $(\Omega = \mathbb{R}^{[0,\infty)}, \mathcal{F} = \mathcal{B}(\mathbb{R})^{[0,\infty)}, \mathbb{P})$ has properties (i,ii,iii) of Definition 1.1.3.

**Proof.** The proof could contain the following ingredients.

(1) Show that for $0 \leq t_0 < t_1 < \cdots < t_n$, there exist multivariate normal distributions with covariance matrices

$$\Sigma = \begin{pmatrix}
    t_0 & 0 & \cdots & 0 \\
    0 & t_1 - t_0 & \cdots & 0 \\
    0 & 0 & t_2 - t_1 & \cdots & 0 \\
    \vdots & \vdots & \ddots & \ddots & \vdots \\
    0 & 0 & 0 & \cdots & t_n - t_{n-1}
\end{pmatrix},$$

and

$$\Sigma_{t_0,\ldots,t_n} = \begin{pmatrix}
    t_0 & t_0 & \cdots & t_0 \\
    t_0 & t_1 & \cdots & t_1 \\
    t_0 & t_1 & t_2 & \cdots & t_2 \\
    \vdots & \vdots & \ddots & \ddots & \vdots \\
    t_0 & t_1 & t_2 & \cdots & t_n
\end{pmatrix}.$$

(2) Show that a stochastic process $W$ has properties (i,ii,iii) if and only if for all $n \in \mathbb{Z}_+$, $0 \leq t_0 < \ldots < t_n$ the vector $(W_{t_0}, W_{t_1} - W_{t_0}, \ldots, W_{t_n} - W_{t_{n-1}})$ $\overset{d}{=} \mathcal{N}(0, \Sigma)$.

(3) Show that for a stochastic process $W$ the (a) and (b) below are equivalent:

a) for all $n \in \mathbb{Z}_+$, $0 \leq t_0 < \ldots < t_n$ the vector $(W_{t_0}, W_{t_1} - W_{t_0}, \ldots, W_{t_n} - W_{t_{n-1}}) \overset{d}{=} \mathcal{N}(0, \Sigma)$ ;

b) for all $n \in \mathbb{Z}_+$, $0 \leq t_0 < \ldots < t_n$ the vector $(W_{t_0}, W_{t_1}, \ldots, W_{t_n}) \overset{d}{=} \mathcal{N}(0, \Sigma_{t_0,\ldots,t_n})$.

QED

The drawback of Kolmogorov’s Consistency Theorem is, that in principle all functions on the positive real line are possible sample paths. Our aim is the show that we may restrict to the subset of continuous paths in the Brownian motion case.

However, the set of continuous paths is not even a measurable subset of $\mathcal{B}(\mathbb{R})^{[0,\infty)}$, and so the probability that the process $W$ has continuous paths is not well defined. The next section discussed how to get around the problem concerning continuous paths.
1.3 Kolmogorov’s continuity criterion

Why do we really insist on Brownian motion to have continuous paths? First of all, the connection with the physical process. Secondly, without regularity properies like continuity, or weaker right-continuity, events of interest are not ensured to measurable sets. An example is: \( \{ \sup_{t \geq 0} W_t \leq x \}, \inf\{ t \geq 0 | W_t = x \} \).

The idea to address this problem, is to try to modify the constructed process \( W \) in such a way that the resulting process, \( W \) say, has continuous paths and satisfies properties (i,ii,iii), in other words, it has the same fdd’s as \( W \). To make this idea precise, we need the following notions.

**Definition 1.3.1** Let \( X \) and \( Y \) be two stochastic processes, indexed by the same set \( T \) and with the same state space \( (E, \mathcal{E}) \), defined on probability spaces \( (\Omega, \mathcal{F}, P) \) and \( (\Omega', \mathcal{F}', P') \) respectively. The processes are called *versions* of each other, if they have the same fdd’s. In other words, if for all \( n \in \mathbb{Z}_+ \), \( t_1, \ldots, t_n \in T \) and \( B_1, \ldots, B_n \in \mathcal{E} \)

\[
P\{X_{t_1} \in B_1, X_{t_2} \in B_2, \ldots, X_{t_n} \in B_n\} = P\{Y_{t_1} \in B_1, Y_{t_2} \in B_2, \ldots, Y_{t_n} \in B_n\}.
\]

\( X \) and \( Y \) are both \( (E, \mathcal{E}) \)-valued stochastic processes. They can be viewed as random variables with values in the measurable path space \( (E^T, \mathcal{E}^T) \). \( X \) induces a probability measure \( P_X \) on the path space with \( P_X\{A\} = P\{X^{-1}(A)\} \). In the same way \( Y \) induces a probability \( P_Y \) on the path space. Since \( X \) and \( Y \) have the same fdds, it follows for each \( n \in \mathbb{Z}_+ \) and \( t_1 < \cdots < t_n, t_1, \ldots, t_n \in T \), and \( A_1, \ldots, A_n \in \mathcal{E} \) that

\[
P_X\{A\} = P_Y\{A\},
\]

for \( A = \{x \in E^T | x_{t_i} \in A_i, i = 1, \ldots, n\} \). The collection of sets \( B \) of this form are a \( \pi \)-system generating \( \mathcal{E}^T \) (cf. remark after BN Lemma 2.1), hence \( P_X = P_Y \) on \( (E^T, \mathcal{E}^T) \) by virtue of BN Lemma 1.2(i).

**Definition 1.3.2** Let \( X \) and \( Y \) be two stochastic processes, indexed by the same set \( T \) and with the same state space \( (E, \mathcal{E}) \), defined on the same probability space \( (\Omega, \mathcal{F}, P) \).

i) The processes are called *modifications* of each other, if for every \( t \in T \)

\[
X_t = Y_t, \quad \text{a.s.}
\]

ii) The processes are called *indistinguishable*, if there exists a set \( \Omega^* \in \mathcal{F} \), with \( P\{\Omega^*\} = 1 \), such that for every \( \omega \in \Omega^* \) the paths \( t \to X_t(\omega) \) and \( t \to Y_t(\omega) \) are equal.

The third notion is stronger than the second notion, which in turn is clearly stronger than the first one: if processes are indistinguishable, then they are modifications of each other. If they are modifications of each other, then they certainly are versions of each other. The reverse is not true in general (cf. Exercises 1.3, 1.6). The following theorem gives a sufficient condition for a process to have a continuous modification. This condition (1.3.1) is known as Kolmogorov’s continuity condition.

Denote by \( C^d[0,T] \) the collection of \( \mathbb{R}^d \)-valued continuous functions on \([0,T]\).
Theorem 1.3.3 (Kolmogorov’s continuity criterion) Let $X = (X_t)_{t \in [0,T]}$ be an $(\mathbb{R}^d, \mathcal{B}^d)$-valued stochastic process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose that there exist constants $\alpha, \beta, K > 0$ such that
\[
\mathbb{E}[|X_t - X_s|^\alpha] \leq K|t - s|^{1+\beta},
\] (1.3.1)
for all $s, t \in [0,T]$. Then there exists a (everywhere!) continuous modification $\hat{X}$ of $X$, i.o.w. $\hat{X}(\omega)$ is a continuous function on $[0,T]$ for each $\omega \in \Omega$. Thus the map $\hat{X} : (\Omega, \mathcal{F}, \mathbb{P}) \to (C^d[0,T], C^d[0,T] \cap \mathcal{B}(\mathbb{R}^d)[0,T])$ is an $\mathcal{F}/C^d[0,T] \cap \mathcal{B}(\mathbb{R}^d)[0,T]$-measurable map.

Note: $\beta > 0$ is needed for the continuous modification to exist. See Exercise 1.5.

Proof. The proof consists of the following steps:

1. (1.3.1) implies that $X_t$ is continuous in probability on $[0,T]$;
2. $X_t$ is a.s. uniformly continuous on a countable, dense subset $D \subset [0,T]$;
3. ‘Extend’ $X$ to a continuous process $Y$ on all of $[0,T]$.
4. Show that $Y$ is a well-defined stochastic process, and a continuous modification of $X$.

Without loss of generality we may assume that $T = 1$.

Step 1 Apply Chebychev’s inequality to the r.v. $Z = |X_t - X_s|$ and the function $\phi : \mathbb{R} \to \mathbb{R}^+$ given by
\[
\phi(x) = \begin{cases} 0, & x \leq 0 \\ x^\alpha, & x > 0. \end{cases}
\]
Since $\phi$ is non-decreasing, non-negative and $\mathbb{E}\phi(Z) < \infty$, it follows for every $\epsilon > 0$ that
\[
\mathbb{P}\{|X_t - X_s| > \epsilon\} \leq \frac{\mathbb{E}[|X_t - X_s|^\alpha]}{\epsilon^\alpha} \leq \frac{K|t - s|^{1+\beta}}{\epsilon^\alpha}. \tag{1.3.2}
\]
Let $t, t_1, \ldots \in [0,1]$ with $t_n \to t$ as $n \to \infty$. By the above,
\[
\lim_{n \to \infty} \mathbb{P}\{|X_{t_n} - X_t| > \epsilon\} = 0,
\]
for any $\epsilon > 0$. Hence $X_{t_n} \overset{P}{\to} X_t$, $n \to \infty$. In other words, $X_t$ is continuous in probability.

Step 2 As the set $D$ we choose the dyadic rationals. Let $D_n = \{k/2^n \mid k = 0, \ldots, 2^n\}$. Then $D_n$ is an increasing sequence of sets. Put $D = \cup_n D_n = \lim_{n \to \infty} D_n$. Clearly $\bar{D} = [0,1]$, i.e. $D$ is dense in $[0,1]$.

Fix $\gamma \in (0, \beta/\alpha)$. Apply Chebychev’s inequality (1.3.2) to obtain
\[
\mathbb{P}\{|X_{k/2^n} - X_{(k-1)/2^n}| > 2^{-\gamma n}\} \leq \frac{K2^{-n(1+\beta)}}{2^{-\gamma n \alpha}} = K2^{-n(1+\beta-\alpha \gamma)}.
\]
It follows that
\[
\mathbb{P}\left\{ \max_{1 \leq k \leq 2^n} |X_{k/2^n} - X_{(k-1)/2^n}| > 2^{-\gamma n}\right\} \leq 2^n \mathbb{P}\{|X_{k/2^n} - X_{(k-1)/2^n}| > 2^{-\gamma n}\} \leq 2^n K2^{-n(1+\beta-\alpha \gamma)} = K2^{-n(\beta-\alpha \gamma)}.
\]
Define the set \( A_n = \{ \max_{1 \leq k \leq 2^n} |X_{k/2^n} - X_{(k-1)/2^n}| > 2^{-\gamma n} \} \). Then

\[
\sum_n P(A_n) \leq \sum_n K2^{-n(\beta - \alpha \gamma)} = K \frac{1}{1 - 2^{-(\beta - \alpha \gamma)}} < \infty,
\]

since \( \beta - \alpha \gamma > 0 \). By virtue of the first Borel-Cantelli Lemma this implies that \( P(\lim \sup_m A_m) = P(\bigcap_{m=1}^\infty \cup_{n=m}^\infty A_n) = 0 \). Hence there exists a set \( \Omega^* \subset \Omega \), \( \Omega^* \in \mathcal{F} \), with \( P(\Omega^*) = 1 \), such that for each \( \omega \in \Omega^* \) there exists \( N_\omega \), for which \( \omega \notin \cup_{n \geq N_\omega} A_n \), in other words

\[
\max_{1 \leq k \leq 2^n} \|X_{k/2^n}(\omega) - X_{(k-1)/2^n}(\omega)\| \leq 2^{-\gamma n}, \quad n \geq N_\omega. \tag{1.3.3}
\]

Fix \( \omega \in \Omega^* \). We will show the existence of a constant \( K' \), such that

\[
\|X_t(\omega) - X_s(\omega)\| \leq K'|t - s|^{\gamma}, \quad \forall s, t \in D, 0 < t - s < 2^{-N_\omega}. \tag{1.3.4}
\]

Indeed, this implies uniform continuity of \( X_t(\omega) \) for \( t \in D \), for \( \omega \in \Omega^* \). Step 2 will then be proved.

Let \( s, t \) satisfy \( 0 < t - s < 2^{-N_\omega} \). Hence, there exists \( n \geq N_\omega \), such that \( 2^{-(n+1)} \leq t - s < 2^{-n} \).

Fix \( n \geq N_\omega \). For the moment, we restrict to the set of \( s, t \in \cup_{m \geq n+1} D_m \), with \( 0 < t - s < 2^{-n} \). By induction to \( m \geq n + 1 \) we will first show that

\[
\|X_t(\omega) - X_s(\omega)\| \leq 2 \sum_{k=n+1}^m 2^{-\gamma k}, \tag{1.3.5}
\]

if \( s, t \in D_m \).

Suppose that \( s, t \in D_{n+1} \). Then \( t - s = 2^{-(n+1)} \). Thus \( s, t \) are neighbouring points in \( D_{n+1} \), i.e. there exists \( k \in \{0, \ldots, 2^{n+1} - 1\} \), such that \( t = k/2^{n+1} \) and \( s = (k + 1)/2^{n+1} \). (1.3.5) with \( m = n + 1 \) follows directly from (1.3.3). Assume that the claim holds true upto \( m \geq n + 1 \). We will show its validity for \( m + 1 \).

Put \( s' = \min\{u \in D_m \mid u \geq s\} \) and \( t' = \max\{u \in D_m \mid u \leq t\} \). By construction \( s \leq s' \leq t' \leq t \), and \( s' - s, t - t' \leq 2^{-(m+1)} \). Then \( 0 < t' - s' \leq t - s < 2^{-n} \). Since \( s', t' \in D_m \), they satisfy the induction hypothesis. We may now apply the triangle inequality, (1.3.3) and the induction hypothesis to obtain

\[
\|X_t(\omega) - X_s(\omega)\| \leq \|X_t(\omega) - X_{t'}(\omega)\| + \|X_{t'}(\omega) - X_{s'}(\omega)\| + \|X_{s'}(\omega) - X_s(\omega)\| \leq 2^{-\gamma(m+1)} + 2 \sum_{k=n+1}^m 2^{-\gamma k} + 2^{-\gamma(m+1)} = 2 \sum_{k=n+1}^m 2^{-\gamma k}.
\]

This shows the validity of (1.3.5). We prove (1.3.4). To this end, let \( s, t \in D \) with \( 0 < t - s < 2^{-N_\omega} \). As noted before, there exists \( n > N_\omega \), such that \( 2^{-(n+1)} \leq t - s < 2^{-n} \). Then there exists \( m \geq n + 1 \) such that \( t, s \in D_m \). Apply (1.3.5) to obtain

\[
\|X_t(\omega) - X_s(\omega)\| \leq 2 \sum_{k=n+1}^m 2^{-\gamma k} \leq \frac{2}{1 - 2^{-\gamma}} 2^{-\gamma(n+1)} \leq \frac{2}{1 - 2^{-\gamma}}|t - s|^{\gamma}.
\]
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Consequently (1.3.4) holds with constant $K' = 2/(1 - 2^{-\gamma})$.

**Step 3** Define a new stochastic process $Y = (Y_t)_{t \in [0,1]}$ on $(\Omega, \mathcal{F}, P)$ as follows: for $\omega \notin \Omega^*$, we put $Y_t = 0$ for all $t \in [0,1]$; for $\omega \in \Omega^*$ we define

$$Y_t(\omega) = \begin{cases} X_t(\omega), & \text{if } t \in D, \\ \lim_{t_n \to t, t_n \in D} X_{t_n}(\omega), & \text{if } t \notin D. \end{cases}$$

For each $\omega \in \Omega^*$, $t \to X_t(\omega)$ is uniformly continuous on the dense subset $D$ of $[0,1]$. It is a theorem from Analysis, that $t \to X_t(\omega)$ can be uniquely extended as a continuous function on $[0,1]$. This is the function $t \to Y_t(\omega)$, $t \in [0,1]$.

**Step 4** Uniform continuity of $X$ implies that $Y$ is a well-defined stochastic process. Since $X$ is continuous in probability, it follows that $Y$ is a modification of $X$ (Exercise 1.4). See BN §5 for a useful characterisation of convergence in probability.

The fact that Kolmogorov’s continuity criterion requires $K' |t - s|^{1+\beta}$ for some $\beta > 0$, guarantees uniform continuity of a.a. paths $X(\omega)$ when restricted to the dyadic rationals, whilst it does not so for $\beta = 0$ (see Exercise 1.5). This uniform continuity property is precisely the basis of the proof of the Criterion.

**Corollary 1.3.4** Brownian motion exists.

**Proof.** By Corollary 1.2.4 there exists a process $W = (W_t)_{t \geq 0}$ that has properties (i,ii,iii) of Definition 1.1.3. By property (iii) the increment $W_t - W_s$ has a $N(0,t-s)$-distribution for all $s \leq t$. This implies that $\mathbb{E}(W_t - W_s)^4 = (t-s)^2 \mathbb{E}Z^4$, with $Z$ a standard normally distributed random variable. This means the Kolmogorov’s continuity condition (1.3.1) is satisfied with $\alpha = 4$ and $\beta = 1$. So for every $T \geq 0$, there exists a continuous modification $W^T = (W^T_t)_{t \in [0,T]}$ of the process $(W_t)_{t \in [0,T]}$. Now define the process $X = (X_t)_{t \geq 0}$ by

$$X_t = \sum_{n=1}^{\infty} W^n_t \mathbf{1}_{[\gamma n, n)}(t).$$

In Exercise 1.7 you are asked to show that $X$ is a Brownian motion process.

**Remarks on the canonical process** Theorem 1.3.3 has allowed us to define Brownian motion as a canonical stochastic process with everywhere continuous paths.

The theorem has allowed us to ‘kick out’ the discontinuous paths from the underlying space. In fact, what we have implicitly shown, is that the ‘outer measure’ of the set of continuous paths is 1 by a complicated argument. The lemma below provides a general result allowing to restrict to a set with outer measure 1.
Let $(\Omega, \mathcal{F}, P)$ be a probability space. Define the outer measure

$$P^*\{A\} = \inf_{B \in \mathcal{F}, B \supseteq A} P\{B\}.$$ 

Lemma 1.3.5 Suppose that $A$ is a subset of $\Omega$ with $P^*\{A\} = 1$. Then for any $F \in \mathcal{F}$, one has $P^*\{F\} = P\{F\}$. Moreover, $(A, A, P^*)$ is a probability space, where $A = \{A \cap F | F \in \mathcal{F}\}$.

A second possibility is the construction mentioned at the end of section 1.1, described in more generality below. Given any $(E, \mathcal{E})$-valued stochastic process $X$ on an underlying probability space $(\Omega, \mathcal{F}, P)$. Then $X : (\Omega, \mathcal{F}) \to (E^T, \mathcal{E}^T)$ is a measurable map inducing a probability measure $P_X$ on the path space $(E^T, \mathcal{E}^T)$. The canonical map on $(E^T, \mathcal{E}^T, P_X)$ now has the same distribution as $X$ by construction. Hence, we can always associate a canonical stochastic process with a given stochastic process.

Suppose now that there exists a subset $\Gamma \subset E^T$, such that $X : \Omega \to \Gamma \cap E^T$. That is, the paths of $X$ have a certain structure. Then $X$ is $\mathcal{F}/\Gamma \cap \mathcal{E}^T$-measurable, and induces a probability measure $P_X$ on $(\Gamma, \Gamma \cap \mathcal{E}^T)$. Again, we may consider the canonical process on this restricted probability space $(\Gamma, \Gamma \cap \mathcal{E}^T, P_X)$.

1.4 Gaussian processes

Brownian motion is an example of a so-called Gaussian process. The general definition is as follows.

Definition 1.4.1 A real-valued stochastic process is called Gaussian of all its fdd’s are Gaussian, in other words, if they are multivariate normal distributions.

Let $X$ be a Gaussian process indexed by the set $T$. Then $m(t) = \mathbb{E}X_t$, $t \in T$, is the mean function of the process. The function $r(s, t) = \text{cov}(X_s, X_t)$, $(s, t) \in T \times T$, is the covariance function. By virtue of the following uniqueness lemma, fdd’s of Gaussian processes are determined by their mean and covariance functions.

Lemma 1.4.2 Two Gaussian processes with the same mean and covariance functions are versions of each other.

Proof. See Exercise 1.8. QED

Brownian motion is a special case of a Gaussian process. In particular it has $m(t) = 0$ for all $t \geq 0$ and $r(s, t) = s \wedge t$, for all $s \leq t$. Any other Gaussian process with the same mean and covariance function has the same fdd’s as BM itself. Hence, it has properties (i,ii,iii) of Definition 1.1.3. We have the following result.

Lemma 1.4.3 A continuous or a.s. continuous Gaussian process $X = (X_t)_{t \geq 0}$ is a BM process if and only if it has the same mean function $m(t) = \mathbb{E}X_t = 0$ and covariance function $r(s, t) = \mathbb{E}X_sX_t = s \wedge t$. 
The lemma looks almost trivial, but provides us with a number extremely useful scaling and symmetry properties of BM!

Remark that a.s. continuity means that the collection of discontinuous paths is contained in a null-set. By continuity we mean that all paths are continuous.

**Theorem 1.4.4** Let \( W \) be a BM process on \((\Omega, \mathcal{F}, P)\). Then the following are BM processes as well:

i) **time-homogeneity** for every \( s \geq 0 \) the shifted process \( W(s) = (W_{t+s} - W_s)_{t \geq 0} \);

ii) **symmetry** the process \(-W = (-W_t)_{t \geq 0} \);

iii) **scaling** for every \( a > 0 \), the process \( W^a \) defined by \( W^a_t = a^{-1/2} W_{at} \);

iv) **time inversion** the process \( X = (X_t)_{t \geq 0} \) with \( X_0 = 0 \) and \( X_t = tW_{1/t}, \ t > 0 \).

If \( W \) has (a.s.) continuous paths then \( W(s) \), \(-W\) and \( W^a \) have (a.s.) continuous paths and \( X \) has a.s. continuous paths. There exists a set \( \Omega^* \in \mathcal{F} \), such that \( X \) has continuous paths on \((\Omega^*, \mathcal{F} \cap \Omega^*, P)\).

**Proof.** We would like to apply Lemma 1.4.3. To this end we have to check that (i) the defined processes are Gaussian; (ii) that (almost all) sample paths are continuous and (iii) that they have the same mean and covariance functions as BM. In Exercise 1.9 you are asked to show this for the processes in (i,ii,iii). We will give an outline of the proof (iv).

The most interesting step is to show that almost all sample paths of \( X \) are continuous. The remainder is analogous to the proofs of (i,ii,iii).

So let us show that almost all sample paths of \( X \) are continuous. By time inversion, it is immediate that \((X_t)_{t \geq 0}\) (a.s.) has continuous sample paths if \( W \) has. We only need show a.s. continuity at \( t = 0 \), that is, we need to show that \( \lim_{t \downarrow 0} X_t = 0, \) a.s.

Let \( \Omega^* = \{ \omega \in \Omega | (W_t(\omega))_{t \geq 0} \text{ continuous}, W_0(\omega) = 0 \} \). By assumption \( \Omega \setminus \Omega^* \) is contained in a \( P \)-null set. Further, \( (X_t)_{t \geq 0} \) has continuous paths on \( \Omega^* \).

Then \( \lim_{t \downarrow 0} X_t(\omega) = 0 \) iif for all \( \epsilon > 0 \) there exists \( \delta_\omega > 0 \) such that \(|X_t(\omega)| < \epsilon \) for all \( t \leq \delta_\omega \). This is true if and only if for all integers \( m \geq 1 \), there exists an integer \( n_\omega \), such that \(|X_q(\omega)| < 1/m \) for all \( q \in \mathbb{Q} \) with \( q < 1/n_\omega \), because of continuity of \( X_t(\omega), \ t > 0 \). Check that this implies

\[
\{ \omega : \lim_{t \downarrow 0} X_t(\omega) = 0 \} \cap \Omega^* = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{q \in (0,1/n]} \{ \omega : |X_q(\omega)| < 1/m \} \cap \Omega^*.
\]

The fdd’s of \( X \) and \( W \) are equal. Hence (cf. Exercise 1.10)

\[
P\{ \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{q \in (0,1/n]} \{ \omega : |X_q(\omega)| < 1/m \} \} = P\{ \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{q \in (0,1/n]} \{ \omega : |W_q(\omega)| < 1/m \} \}.
\]

It follows that (cf. Exercise 1.10) the probability of the latter equals 1. As a consequence

\[
P\{ \omega : \lim_{t \downarrow 0} X_t(\omega) = 0 \} = 1.
\]

QED

These scaling and symmetry properties can be used to show a number of properties of Brownian motion. The first is that Brownian motion sample paths oscillate between \(+\infty\) and \(-\infty\).
Corollary 1.4.5 Let $W$ be a BM with the property that all paths are continuous. Then
\[ P\{\sup_{t \geq 0} W_t = \infty, \inf_{t \geq 0} W_t = -\infty\} = 1. \]

Proof. It is sufficient to show that
\[ P\{\sup_{t \geq 0} W_t = \infty\} = 1. \]  \hfill (1.4.1)

Indeed, the symmetry property implies
\[ \sup_{t \geq 0} W_t \overset{d}{=} \sup_{t \geq 0} (-W_t) = -\inf_{t \geq 0} W_t. \]

Hence (1.4.1) implies that $P\{\inf_{t \geq 0} W_t = -\infty\} = 1$. As a consequence, the probability of the intersection equals 1 (why?).

First of all, notice that $\sup_t W_t$ is well-defined. We need to show that $\sup_t W_t$ is a measurable function. This is true (cf. BN Lemma 1.3) if $\{\sup_t W_t \leq x\}$ is measurable for all $x \in \mathbb{R}$. (Q is sufficient of course).

This follows from
\[ \{\sup_t W_t \leq x\} = \bigcap_{q \in \mathbb{Q}} \{W_q \leq x\}. \]

Here we use that all paths are continuous. We cannot make any assertions on measurability of $\{W_q \leq x\}$ restricted to the set of discontinuous paths, unless $\mathcal{F}$ is $\mathcal{P}$-complete.

By the scaling property we have for all $a > 0$
\[ \sup_t W_t \overset{d}{=} \sup_{t} \frac{1}{\sqrt{a}} W_{at} = \frac{1}{\sqrt{a}} \sup_t W_t. \]

It follows for $n \in \mathbb{Z}_+$ that
\[ P\{\sup_t W_t \leq n\} = P\{n^2 \sup_t W_t \leq n\} = P\{\sup_t W_t \leq 1/n\}. \]

By letting $n$ tend to infinity, we see that
\[ P\{\sup_t W_t < \infty\} = P\{\sup_t W_t \leq 0\}. \]

Thus, for (1.4.1) it is sufficient to show that $P\{\sup_t W_t \leq 0\} = 0$. We have
\[ P\{\sup_t W_t \leq 0\} \leq P\{W_1 \leq 0, \sup_{t \geq 1} W_t \leq 0\} \leq P\{W_1 \leq 0, \sup_{t \geq 1} W_t - W_1 < \infty\} = P\{W_1 \leq 0\} P\{\sup_{t \geq 1} W_t - W_1 < \infty\}, \]

by the independence of Brownian motion increments. By the time-homogeneity of BM, the latter probability equals the probability that the supremum of BM is finite. We have just showed that this equals $P\{\sup W_t \leq 0\}$. And so we find
\[ P\{\sup_t W_t \leq 0\} \leq \frac{1}{2} P\{\sup W_t \leq 0\}. \]
1.5. NON-DIFFERENTIABILITY OF THE BROWNIAN SAMPLE PATHS

This shows that \( P\{\sup_t W_t \leq 0\} = 0 \) and so we have shown (1.4.1).

Since BM has a.s. continuous sample paths, this implies that almost every path visits every point of \( \mathbb{R} \). This property is called recurrence. With probability 1 it even visits every point infinitely often. However, we will not further pursue this at the moment and merely mention the following statement.

**Corollary 1.4.6** BM is recurrent.

An interesting consequence of the time inversion property is the following strong law of large numbers for BM.

**Corollary 1.4.7** Let \( W \) be a BM. Then

\[
\frac{W_t}{t} \xrightarrow{a.s.} 0, \quad t \to \infty.
\]

**Proof.** Let \( X \) be as in part (iv) of Theorem 1.4.4. Then

\[
P\{\frac{W_t}{t} \to 0, \quad t \to \infty\} = P\{X_{1/t} \to 0, \quad t \to \infty\} = 1.
\]

QED

1.5 Non-differentiability of the Brownian sample paths

We have already seen that the sample paths of \( W \) are continuous functions that oscillate between \( +\infty \) and \( -\infty \). Figure 1.1 suggests that the sample paths are very rough. The following theorem shows that this is indeed the case.

**Theorem 1.5.1** Let \( W \) be a BM defined on the space \( (\Omega, \mathcal{F}, P) \). There is a set \( \Omega^* \) with \( P\{\Omega^*\} = 1 \), such that the sample path \( t \mapsto W(\omega) \) is nowhere differentiable, for any \( \omega \in \Omega^* \).

**Proof.** Let \( W \) be a BM. Consider the upper and lower right-hand derivatives

\[
D^W(t, \omega) = \limsup_{h \downarrow 0} \frac{W_{t+h}(\omega) - W_t(\omega)}{h},
\]

\[
D_W(t, \omega) = \liminf_{h \downarrow 0} \frac{W_{t+h}(\omega) - W_t(\omega)}{h}.
\]

Let

\[ A = \{ \omega \mid \text{there exists } t \geq 0 \text{ such that } D^W(t, \omega) \text{ and } D_W(t, \omega) \text{ are finite } \}. \]

Note that \( A \) is not necessarily a measurable set. We will therefore show that \( A \) is contained in a measurable set \( B \) with \( P\{B\} = 0 \). In other words, \( A \) has outer measure 0.

To define the set \( B \), first consider for \( k, n \in \mathbb{Z}_+ \) the random variable

\[
X_{n,k} = \max \left\{ |W_{(k+1)/2^n} - W_{k/2^n}|, |W_{(k+2)/2^n} - W_{(k+1)/2^n}|, |W_{(k+3)/2^n} - W_{(k+2)/2^n}| \right\}.
\]
Define for $n \in \mathbb{Z}_+$

$$Y_n = \min_{k \leq n/2^n} X_{n,k}.$$ 

A the set $B$ we choose

$$B = \bigcap_{n=1}^{\infty} \bigcap_{k=1}^{\infty} \{Y_k \leq k2^{-k}\}.$$

We claim that $A \subseteq B$ and $P\{B\} = 0$.

To prove the inclusion, let $\omega \in A$. Then there exists $t = t_\omega$, such that $D_W(t, \omega), D^W(t, \omega)$ are finite. Hence, there exists $K = K_\omega$, such that

$$-K < D_W(t, \omega) \leq D^W(t, \omega) < K.$$ 

As a consequence, there exists $\delta = \delta_\omega$, such that

$$|W_s(\omega) - W_t(\omega)| \leq K \cdot |s - t|, \quad s \in [t, t + \delta]. \quad (1.5.1)$$

Now take $n = n_\omega \in \mathbb{Z}_+$ so large that

$$\frac{4}{2^n} < \delta, \quad 8K < n, \quad t < n. \quad (1.5.2)$$

Next choose $k \in \mathbb{Z}_+$, such that

$$\frac{k - 1}{2^n} \leq t < \frac{k}{2^n}. \quad (1.5.3)$$

By the first relation in (1.5.2) we have that

$$\left| \frac{k + 3}{2^n} - t \right| \leq \left| \frac{k + 3}{2^n} - \frac{k - 1}{2^n} \right| \leq \frac{4}{2^n} < \delta,$$

so that $k/2^n, (k + 1)/2^n, (k + 2)/2^n, (k + 3)/2^n \in [t, t + \delta]$. By (1.5.1) and the second relation in (1.5.2) we have our choice of $n$ and $k$ that

$$X_{n,k}(\omega) \leq \max \{ |W_{(k+1)/2^n} - W_t|, |W_t - W_{k/2^n}|, |W_{(k+2)/2^n} - W_t|, |W_t - W_{(k+1)/2^n}|, |W_{(k+3)/2^n} - W_t|, |W_t - W_{(k+2)/2^n}| \}$$

$$\leq 2K \frac{4}{2^n} < \frac{n}{2^n}.$$ 

The third relation in (1.5.2) and (1.5.3) it holds that $k - 1 \leq t2^n < n2^n$. This implies $k \leq n2^n$ and so $Y_n(\omega) \leq X_{n,k}(\omega) \leq n/2^n$, for our choice of $n$.

Summarising, $\omega \in A$ implies that $Y_n(\omega) \leq n/2^n$ for all sufficiently large $n$. This implies $\omega \in B$. We have proved that $A \subseteq B$.

In order to complete the proof, we have to show that $P\{B\} = 0$. Note that $|W_{(k+1)/2^n} - W_{k/2^n}|, |W_{(k+2)/2^n} - W_{(k+1)/2^n}|$ and $|W_{(k+3)/2^n} - W_{(k+2)/2^n}|$ are i.i.d. random variables. We have for any $\epsilon > 0$ and $k = 0, \ldots, n2^n$ that

$$P\{X_{n,k} \leq \epsilon\} \leq P\{|W_{(k+i)/2^n} - W_{(k+i-1)/2^n}| \leq \epsilon, i = 1, 2, 3\}$$

$$\leq (P\{|W_{(k+1)/2^n} - W_{(k+2)/2^n}| \leq \epsilon\})^3 = (P\{|W_{1/2^n}| \leq \epsilon\})^3$$

$$= (P\{|W_1| \leq 2^{n/2}\epsilon\})^3 \leq (2 \cdot 2^{n/2}\epsilon)^3 = 2^{3n/2+1}\epsilon^3.$$
1.6 Filtrations and stopping times

If \( W \) is a BM, the increment \( W_{t+h} - W_t \) is independent of ‘what happened up to time \( t \).’ In this section we introduce the concept of a filtration to formalise the notion of ‘information that we have up to time \( t \).’ The probability space \((\Omega, \mathcal{F}, \mathbb{P})\) is fixed again and we suppose that \( T \) is a subinterval of \( \mathbb{Z}_+ \) or \( \mathbb{R}_+ \).

**Definition 1.6.1** A collection \((\mathcal{F}_t)_{t \in T}\) of sub-\(\sigma\)-algebras is called a filtration if \( \mathcal{F}_s \subseteq \mathcal{F}_t \) for all \( s \leq t \). A stochastic process \( X \) defined on \((\Omega, \mathcal{F}, \mathbb{P})\) and indexed by \( T \) is called adapted to the filtration if for every \( t \in T \), the random variable \( X_t \) is \( \mathcal{F}_t \)-measurable. Then \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in T}, \mathbb{P})\) is a filtered probability space.

We can think of a filtration as a flow of information. The \(\sigma\)-algebra \( \mathcal{F}_t \) contains the events that can happen ‘upto time \( t \).’ An adapted process is a process that ‘does not look into the future’. If \( X \) is a stochastic process, then we can consider the filtration \((\mathcal{F}_t^X)_{t \in T}\) generated by \( X \):

\[
\mathcal{F}_t^X = \sigma(X_s, s \leq t).
\]

We call this the filtration generated by \( X \), or the natural filtration of \( X \). It is the ‘smallest’ filtration, to which \( X \) is adapted. Intuitively, the natural filtration of a process keeps track of the ‘history’ of the process. A stochastic process is always adapted to its natural filtration.

**Canonical process and filtration** If \( X \) is a canonical process on \((\mathcal{E}, \mathbb{E}T)\) with \( \mathcal{E}T \), then \( \mathcal{F}_t^X = \mathcal{E}[0,t] \cap \mathbb{E}T \).

As has been pointed out in Section 1.3, with a stochastic process \( X \) one can associate a canonical process with the same distribution.

Indeed, suppose that \( X : \Omega \to \Gamma \cap E^T \). The canonical process on \( \Gamma \cap E^T \) is adapted to the filtration \((\mathcal{E}[0,t] \cap \Gamma)_t\).

Review BN §2, the paragraph on \(\sigma\)-algebra generated by a random variable or a stochastic process.

If \((\mathcal{F}_t)_{t \in T}\) is a filtration, then for \( t \in T \) we may define the \(\sigma\)-algebra

\[
\mathcal{F}_{t+} = \bigcap_{n=1}^{\infty} \mathcal{F}_{t+1/n}.
\]
This is the $\sigma$-algebra $\mathcal{F}_t$, augmented with the events that ‘happen immediately after time $t’$. The collection $(\mathcal{F}_t)_{t \in T}$ is again a filtration (see Exercise 1.16). Cases in which it coincides with the original filtration are of special interest.

**Definition 1.6.2** We call a filtration $(\mathcal{F}_t)_{t \in T}$ right-continuous if $\mathcal{F}_{t+} = \mathcal{F}_t$ for all $t \in T$.

Intuitively, right-continuity of a filtration means that ‘nothing can happen in an infinitesimal small time-interval’ after the observed time instant. Note that for every filtration $(\mathcal{F}_t)$, the corresponding filtration $(\mathcal{F}_{t+})$ is always right-continuous.

In addition to right-continuity it is often assumed that $\mathcal{F}_0$ contains all events in $\mathcal{F}_\infty$ that have probability 0, where

$$\mathcal{F}_\infty = \sigma(\mathcal{F}_t, t \geq 0).$$

As a consequence, every $\mathcal{F}_t$ then also contains these events.

**Definition 1.6.3** A filtration $(\mathcal{F}_t)_{t \in T}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to satisfy the usual conditions if it is right-continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-negligible events of $\mathcal{F}_\infty$.

**Stopping times** We now introduce a very important class of ‘random times’ that can be associated with a filtration.

**Definition 1.6.4** An $[0, \infty]$-valued random variable $\tau$ is called a $(\mathcal{F}_t)_t$-stopping time, if for every $t \in T$ it holds that the event $\{\tau \leq t\}$ is $\mathcal{F}_t$-measurable. We may drop the dependence on the filtration in the notation, whenever filtration at play is clear.

If $\tau < \infty$, we call $\tau$ a finite stopping time, and if $\mathbb{P}\{\tau < \infty\} = 1$, then we call $\tau$ a.s. finite. Similarly, if there exists a constant $K$ such that $\tau(\omega) \leq K$, then $\tau$ is said to be bounded, and if $\mathbb{P}\{\tau \leq K\} = 1$ $\tau$ is a.s. bounded.

Loosely speaking, $\tau$ is a stopping time if for every $t \in T$ we can determine whether $\tau$ has occurred before time $t$ on basis of the information that we have upto time $t$. Note that $\tau$ is $\mathcal{F}_t/\mathcal{B}([0, \infty])$-measurable.

With a stopping time $\tau$ we can associate the the $\sigma$-algebra $\sigma^\tau$ generated by $\tau$. However, this $\sigma$-algebra only contains the information about when $\tau$ occurred. If $\tau$ is associated with an adapted process $X$, then $\sigma^\tau$ contains no further information on the history of the process upto the stopping time. For this reason we associate with $\tau$ the (generally) larger $\sigma$-algebra $\mathcal{F}_\tau$ defined by

$$\mathcal{F}_\tau = \{A \in \mathcal{F}_\infty : A \cap \{\tau \leq t\} \in \mathcal{F}_t \text{ for all } t \in T\}.$$

(see Exercise 1.17). This should be viewed as the collection of all events that happen prior to the stopping time $\tau$. Note that the notation is unambiguous, since a deterministic time $t \in T$ is clearly a stopping time and its associated $\sigma$-algebra is simply the $\sigma$-algebra $\mathcal{F}_t$.

Our loose description of stopping times and stopped $\sigma$-algebra can made more rigorous, when we consider the canonical process with natural filtration.

**Lemma 1.6.5** Let $\Omega \subset E^T$ have the property that for each $t \in T$ and $\omega \in \Omega$ there exists $\overline{\omega} \in \Omega$ with $\overline{\omega}_s = \omega_{s\wedge t}$ for all $s \in T$. Let $\mathcal{F} = E^T \cap \Omega$. Let $X$ be the canonical process on $(\Omega, \mathcal{F})$. Define $\mathcal{F}_s^X = \sigma(X_t, t \in T)$. The following assertions are true.

a) Let $A \subset \Omega$. Then $A \in \mathcal{F}_s^X$ if and only if (i) $A \in \mathcal{F}_s^\infty$ and (ii) $\omega \in A$ and $X_s(\omega) = X_s(\omega')$ for all $s \in T$ with $s \leq t$ imply that $\omega' \in A$. 

b) **Galmarino test** \( \tau \) is an \( (\mathcal{F}_t^X)_{t\geq 0} \)-stopping time if and only if \( \tau \) is \( \mathcal{F}_\infty^X \)-measurable and \( \tau(\omega) \leq t \), \( X_s(\omega) = X_s(\omega') \) for \( s \in T, s \leq t \) implies that \( \tau(\omega') \leq t \), for all \( t \in T \), and \( \omega, \omega' \in \Omega \).

c) Let \( \tau \) be an \( \mathcal{F}_t^X \)-stopping time. Let \( A \subset \Omega \). Then \( A \in \mathcal{F}_{t+}^X \) if and only if (i) \( A \in \mathcal{F}_\infty^X \) and (ii) \( \omega \in A \) and \( X_s(\omega) = X_s(\omega') \) for all \( s \leq \tau(\omega) \) imply that \( \omega' \in A \).

**Proof.** See Exercise 1.24. QED

If the filtration \( (\mathcal{F}_t) \) is right-continuous, then \( \tau \) is a stopping time if and only if \( \{ \tau < t \} \in \mathcal{F}_t \) for all \( t \in T \) (see Exercise 1.23). The latter defines another type of random time called optional time.

**Definition 1.6.6** A \([0, \infty]\)-valued random variable \( \tau \) is called an **optional time** with respect to the filtration \( (\mathcal{F}_t) \) if for every \( t \in T \) it holds that \( \{ \tau < t \} \in \mathcal{F}_t \). If \( \tau < \infty \) almost surely, we call the optional time **finite**.

Check that \( \tau \) is an optional time if and only if \( \tau + t \) is a stopping time for each \( t \in T, t > 0 \).

**Lemma 1.6.7** \( \tau \) is an \( (\mathcal{F}_t)_{t\geq 0} \)-optional time if and only if it is a \( (\mathcal{F}_{t+})_{t\geq 0} \)-stopping time with respect to \( (\mathcal{F}_{t+}) \). Every \( (\mathcal{F}_t)_{t\geq 0} \)-stopping time is a \( (\mathcal{F}_t)_{t\geq 0} \)-optional time.

**Proof.** See Exercise 1.25. QED

The associated \( \sigma \)-algebra \( \mathcal{F}_{\tau+} \) is defined to be
\[
\mathcal{F}_{\tau+} = \{ A \in \mathcal{F}_\infty : A \cap \{ \tau \leq t \} \in \mathcal{F}_{t+}, t \geq 0 \}.
\]

For an \( \{ \mathcal{F}_t \} \)-optional time \( \tau \) it holds (cf. Exercise 1.34) that \( \mathcal{F}_{\tau+} \) is a \( \sigma \)-algebra, with respect to which \( \tau \) is measurable. Moreover \( \mathcal{F}_{\tau+} = \{ A \in \mathcal{F}_\infty : A \cap \{ \tau < t \} \in \mathcal{F}_t, t \geq 0 \} \).

**Special stopping and optional times** The so-called **hitting** and **first entrance times** form an important class of stopping times and optional times. They are related to the first time that the process visits a set \( B \).

**Definition 1.6.8** Let \( X = (X_t)_{t\geq 0} \) an \((E, \mathcal{E})\)-valued stochastic process defined on the underlying probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Let \( B \in \mathcal{E} \). The **first entrance time** of \( B \) is defined by
\[
\sigma_B = \inf \{ t \geq 0 | X_t \in B \}.
\]

The **first hitting time** of \( B \) is defined by
\[
\tau_B = \inf \{ t > 0 | X_t \in B \}.
\]

**Lemma 1.6.9** Let \((E, d)\) be a metric space and let \( \mathcal{B}(E) \) be the Borel-\( \sigma \)-algebra of open sets compatible with the metric \( d \). Suppose that \( X = (X_t)_{t\geq 0} \) is a continuous, \((E, \mathcal{B}(E))\)-valued stochastic process and that \( B \) is closed in \( E \). Then \( \sigma_B \) is an \( (\mathcal{F}_t^X) \)-stopping time.\(^1\)

\(^1\)As is usual, we define \( \inf \emptyset = \infty \).
Proof. Denote the distance of a point $x \in E$ to the set $B$ by $d(x, B)$. In other words

$$d(x, B) = \inf \{d(x, y) \mid y \in B\}.$$ 

First note that $x \to d(x, B)$ is a continuous function. Hence it is $\mathcal{B}(E)$-measurable. It follows that $Y_t = d(X_t, B)$ is $\mathcal{F}_t^X$-measurable as a composition of measurable maps. Since $X_t$ is continuous, the real-valued process $(Y_t)_t$ is continuous as well. Moreover, since $B$ is closed, it holds that $X_t \in B$ if and only if $Y_t = 0$. By continuity of $Y_t$, it follows that $\sigma_B > t$ if and only if $Y_s > 0$ for all $s \leq t$. This means that

$$\{\sigma_B > t\} = \{Y_s > 0, 0 \leq s \leq t\} = \bigcup_{n=1}^{\infty} \bigcap_{q \in Q_t} \{Y_q > \frac{1}{n}\} = \bigcup_{n=1}^{\infty} \bigcap_{q \in Q_t} \{d(X_q, B) > \frac{1}{n}\} \in \mathcal{F}_t^X,$$

where $Q_t = \{tq \mid q \in Q \cap [0, 1]\}$.

QED

**Lemma 1.6.10** Let $(E, d)$ be a metric space and let $\mathcal{B}(E)$ be the Borel-$\sigma$-algebra of open sets compatible with the metric $d$. Suppose that $X = (X_t)_{t \geq 0}$ is a right-continuous, $(E, \mathcal{B}(E))$-valued stochastic process and that $B$ is an open set in $E$. Then, $\tau_B$ is an $(\mathcal{F}_t^X)$-optional time.

Proof. By right-continuity of $X$ and the fact that $B$ is open, $\tau_B(\omega) < t$ if and only if there exists a rational number $0 < q_\omega < t$ such that $X_{q_\omega}(\omega) \in B$. Hence

$$\{\tau_B < t\} = \cup_{q \in (0, t) \cap \mathbb{Q}} \{X_q \in B\}.$$ 

The latter set is $\mathcal{F}_t^X$-measurable, and so is the first.

QED

**Example 1.6.1** Let $W$ be a BM with continuous paths and, for $x > 0$, consider the random variable

$$\tau_x = \inf\{t > 0 \mid W_t = x\}.$$ 

Note that

$$\{\tau_x > t\} = \{W_0 \neq x\} \cap \{\sigma_x > t\}, \quad t > 0.$$ 

Since $W$ is continuous, $\{\sigma_x > t\} \in \mathcal{F}^W_t$, for $t \geq 0$, by Lemma 1.6.9. Additionally, $\{W_0 \neq x\} \in \mathcal{F}^W_0 \subset \mathcal{F}^W_t$, hence $\{\tau_x > t\} \in \mathcal{F}^W_t$, for all $t \geq 0$. We conclude that $\tau_x$ is an $(\mathcal{F}^W_t)_t$-stopping time.

Furthermore, as $\mathbb{P}\{W_0 = 0\} = 1$, it holds that $\mathbb{P}\{\tau_x = \sigma_x\} = 1$. It is therefore sufficient to show that $\{\sigma_x < \infty\}$ is measurable and $\mathbb{P}\{\sigma_x < \infty\} = 1$.

First, $\{\sigma_x < \infty\} = \cup_{n=1}^{\infty} \{\sigma_x \leq n\}$ is a measurable set. Next, consider $A = \{\omega : \sup_{t \geq 0} W_t = \infty, \inf_{t \geq 0} W_t = -\infty\}$. By Corollary 1.4.5, this set has probability 1.

Let $T > |x|$. For each $\omega \in A$, there exist $T_\omega, T'_\omega$, such that $W_{T_\omega} \geq T$, $W_{T'_\omega} \leq -T$. By continuity of paths, there exists $t_\omega \in (T_\omega \wedge T'_\omega, T_\omega \vee T'_\omega)$, such that $W_{t_\omega} = x$. It follows that $A \subset \{\sigma_x < \infty\}$. Hence $\mathbb{P}\{\sigma_x < \infty\} = 1$. Thus, $\mathbb{P}\{\tau_x < \infty\} = 1$.

QED
1.6. FILTRATIONS AND STOPPING TIMES

An important question is whether the first entrance time of a closed set is a stopping time for more general stochastic processes than the continuous ones. The answer in general is that this is not true unless the filtration is suitably augmented with null sets (cf. BN §10). Without augmentation we can derive the two following results. Define $X_{t-} = \liminf_{s \uparrow t} X_s$.

**Lemma 1.6.11** Let $(E,d)$ be a metric space and let $\mathcal{B}(E)$ be the Borel-$\sigma$-algebra of open sets compatible with the metric $d$. Suppose that $X = (X_t)_{t \geq 0}$ is a (everywhere) cadlag, $(E,\mathcal{B}(E))$-valued stochastic process and that $B$ is a closed set in $E$. Then

$$\gamma_B = \inf\{t > 0 \mid X_t \in B, \text{ or } X_{t-} \in B\},$$

is an $(\mathcal{F}_t^X)$-stopping time.

**Lemma 1.6.12** Let $(E,d)$ be a metric space and let $\mathcal{B}(E)$ be the Borel-$\sigma$-algebra of open sets compatible with the metric $d$. Suppose that $X = (X_t)_{t \geq 0}$ is a right-continuous, $(E,\mathcal{B}(E))$-valued stochastic process and that $B$ is closed in $\mathcal{B}(E)$. Let $X$ be defined on the underlying probability space $(\Omega, \mathcal{F}, P)$.

Suppose further that there exist $\mathcal{F}$-measurable random times $0 < \tau_1 < \tau_2 < \ldots$, such that the discontinuities of $X$ are contained in the set $\{\tau_1, \tau_2, \ldots\}$. Then

a) $\tau_1, \tau_2, \ldots$ are $(\mathcal{F}_t^X)$-stopping times;

b) $\sigma_B$ is an $(\mathcal{F}_t^X)$-stopping time.

**Proof.** We prove (a) and check that $\{\tau_1 \leq t\} \in \mathcal{F}_t^X$. Define for $Q_t = \{qt \mid q \in [0,1] \cap \mathbb{Q}\}$

$$G = \bigcup_{m \geq 1} \bigcap_{n \geq 1} \bigcup_{u,s \in Q_t} \{ |u-s| \leq \frac{1}{n}, d(X_u, X_s) > \frac{1}{m} \}.$$ 

**Claim:** $G = \{\tau_1 \leq t\}$.

Let $\omega \in G$. Then there exists $m_\omega$ such that for each $n$ there exists a pair $(u_{n,\omega}, s_{n,\omega})$ with $|u_{n,\omega} - s_{n,\omega}| < 1/n$ for which $d(X_{u_{n,\omega}}, X_{s_{n,\omega}}) > 1/m_\omega$.

If $\omega \notin \{\tau_1 \leq t\}$, then $\tau_1(\omega) > t$, and $s \to X_s(\omega)$ would be continuous on $[0,t]$, hence uniformly continuous. As a consequence, for each for $m_\omega$ there exists $n_\omega$ for which $d(X_s, X_u) < 1/m_\omega$ for $|u-s| < 1/n_\omega$. This contradicts the above, and hence $\omega \in \{\tau_1 \leq t\}$.

To prove the converse, assume that $\omega \in \{\tau_1 \leq t\}$, i.o.w. $s = \tau_1(\omega) \leq t$. By right-continuity this implies that there exists a sequence $t_l \uparrow s$, along which $X_{t_l}(\omega) \not\to X_s(\omega)$. Hence there exists $m$, such that for each $n$ there exists $t_{l(n)}$ with $|s - t_{l(n)}| < 1/n$, for which $d(X_{t_{l(n)}}(\omega), X_s(\omega)) > 1/m$.

By right-continuity, for each $n$ one can find $q_n \geq t_{l(n)}$ and $q \geq s$, $q_n, q \in Q_t$, such that $|q - q_n| < 1/n$ and $d(X_{q(n)}(\omega), X_q(\omega)) > 1/2m$. It follows that

$$\omega \in \bigcap_{n \geq 1} \bigcup_{q,q_n \in Q_t} \{|q - q(n)| < 1/n, d(X_{q(n)}, X_q) > 1/2m\}.$$ 

Hence $\omega \in G$.

To show that $\tau_2$ is a stopping time, we add the requirement $u, s \geq \tau_1$ in the definition of the analogon of the set $G$, etc.
Next we prove (b). We will consider only the case that \( \{\tau_k(\omega)\}_k \) does not have an accumulation point in \([0, t]\) for any \( \omega \in \Omega \). Figure out yourself what to do in the case of an accumulation point! We have to cut out small intervals to the left of jumps. On the remainder we can separate the path \( X_s(\omega), s \leq t \) and the set \( B \), if \( \sigma_B(\omega) > t \). To this end define

\[
I_{k,n} = \{u \mid \tau_k - \frac{1}{n} \leq u < \tau_k \leq t\}.
\]

For each \( \omega \in \Omega \) this is a subset of \([0, t]\). Now, given \( u \in [0, t] \),

\[
\{\omega \mid I_{k,n}(\omega) \ni u\} = \{\omega \mid u < \tau_k(\omega) \leq t \land (u + \frac{1}{n}) \in \mathcal{F}_t^X\}.
\]

Check that

\[
\{\sigma_B > t\} = \bigcap_{n \geq 1} \bigcup_{m \geq 1} \bigcap_{k \in \mathbb{Q}_+} \left( \bigcap_{q \notin I_{k,n}} \{d(X_q, B) > \frac{1}{m}\} \right).
\]

QED

**Measurability of \( X_\tau \) for \( \tau \) an adapted stopping time** We often would like to consider the stochastic process \( X \) evaluated at a finite stopping time \( \tau \). However, it is not a priori clear that the map \( \omega \to X_\tau(\omega) \) is measurable. In other words, that \( X_\tau \) is a random variable. We need measurability of \( X \) in both parameters \( t \) and \( \omega \). This motivates the following definition.

**Definition 1.6.13** An \((E, \mathcal{E})\)-valued stochastic process is called progressively measurable with respect to the filtration \((\mathcal{F}_t)\) if for every \( t \in T \) the map \((s, \omega) \to X_s(\omega)\) is measurable as a map from \(([0, t] \times \Omega, \mathcal{B}([0, t]) \times \mathcal{F}_t)\) to \((E, \mathcal{E})\).

**Lemma 1.6.14** Let \((E, d)\) be a metric space and \( \mathcal{B}(E) \) the Borel-\( \sigma \)-algebra of open sets compatible with \( d \). Every adapted right-continuous, \((E, \mathcal{B}(E))\)-valued stochastic process \( X \) is progressively measurable.

**Proof.** Fix \( t \geq 0 \). We have to check that

\[
\{(s, \omega) \mid X_s(\omega) \in A, s \leq t\} \in \mathcal{B}([0, t]) \times \mathcal{F}_t, \quad \forall A \in \mathcal{B}(E).
\]

For \( n \in \mathbb{Z}_+ \) define the process

\[
X^n_s = \sum_{k=0}^{n-1} X_{(k+1)\frac{t}{n}} 1_{\{(kt/n, (k+1)t/n]\}}(s) + X_0 1_{\{0\}}(s).
\]

This is a measurable process, since

\[
\{(s, \omega) \mid X^n_s(\omega) \in A, s \leq t\} = \bigcup_{k=0}^{n-1} \left( \{s \in (kt/n, (k+1)t/n]\} \times \{\omega \mid X_{(k+1)\frac{t}{n}}(\omega) \in A\} \right) \bigcup \{\{0\} \times \{\omega \mid X_0(\omega) \in A\}\}.
\]

Clearly, \( X^n_s(\omega) \to X_s(\omega), n \to \infty \), for all \((s, \omega) \in [0, t] \times \Omega\), pointwise. By BN Lemma 6.1, the limit is measurable. QED
1.6. FILTRATIONS AND STOPPING TIMES

Review BN §6 containing an example of a non-progressively measurable stochastic process and a stopping time $\tau$ with $X_\tau$ not $\mathcal{F}_\tau$-measurable.

**Lemma 1.6.15** Suppose that $X$ is a progressively measurable process. Let $\tau$ be a stopping time. Then $1_{\{\tau < \infty\}} X_\tau$ is an $\mathcal{F}_\tau$-measurable random element.

**Proof.** We have to show that $\{1_{\{\tau < \infty\}} X_\tau \in B\} \cap \{\tau \leq t\} = \{X_\tau \in B\} \cap \{\tau \leq t\} \in \mathcal{F}_t$, for every $B \in \mathcal{E}$ and every $t \geq 0$. Now note that

$$\{X_\tau \in B\} = \{X_\tau \in B, \tau \leq t\} \cup \{X_t \in B, \tau > t\}.$$

Clearly $\{X_t \in B, \tau > t\} \in \mathcal{F}_t$. If we can show that $\{X_\tau \in B\} \in \mathcal{F}_t$, it easily follows that $\{X_\tau \in B, \tau \leq t\} \in \mathcal{F}_t$. Hence, it suffices to show that the map $\omega \rightarrow X_\tau(\omega)$ is $\mathcal{F}_\tau$-measurable.

To this end consider the map $\phi : ([0, t] \times \Omega, \mathcal{B}([0, t]) \times \mathcal{F}_t) \rightarrow ([0, t] \times \Omega, \mathcal{B}([0, t]) \times \mathcal{F}_t)$ given by $\phi(s, \omega) = (\tau(\omega) \wedge s, \omega)$ is measurable (this is almost trivial, see Exercise 1.26). Using that $X$ is progressively measurable, it follows that the composition map $(s, \omega) \rightarrow X(\phi(s, \omega)) = X_{\tau(\omega)}(\omega)$ is measurable.

By Fubini’s theorem the section map $\omega \rightarrow X_{\tau(\omega)}(\omega)$ is $\mathcal{F}_\tau$-measurable. QED

Very often problems of interest consider a stochastic process upto a given stopping time $\tau$. To this end we define the *stopped process* $X^\tau$ by

$$X_t^\tau = X_{\tau \wedge t} = \begin{cases} X_t, & t < \tau, \\ X_\tau, & t \geq \tau. \end{cases}$$

Using Lemma 1.6.15, and the arguments in the proof, as well as Exercises 1.18 and 1.20, we have the following result.

**Lemma 1.6.16** Let $X$ be progressively measurable with respect to $(\mathcal{F}_t)$ and $\tau$ an $(\mathcal{F}_t)$-stopping time. The following are true.

1. The stopped process $X^\tau$ is progressively measurable with respect to the filtrations $(\mathcal{F}_{\tau \wedge t})_t$ and $(\mathcal{F}_t)_t$.

2. $(\mathcal{F}_{\tau+t})_t$ is a filtration, and the shifted process $\{X_{\tau+t}\}_t$ is progressively measurable with respect to this shifted filtration.

The Lemma is important later on, when we consider processes shifted by a stopping time.

**Proof.** We will first show that $X^\tau$ is progressively measurable with respect to $(\mathcal{F}_{\tau \wedge t})_t$. Since $\mathcal{F}_{\tau \wedge t} \subset \mathcal{F}_t$, progressive measurability with respect to $(\mathcal{F}_t)_t$ immediately follows.

We will use the following statement to characterise some sets that belong to $\mathcal{B}([0, t]) \times \mathcal{F}_{\tau \wedge t}$. Let time $t$ be given, and let $s < t$. For any $C \in \mathcal{B}([0, t]) \times \mathcal{F}_s$, it holds that

$$C \cap \{[0, t] \times \{t \wedge \tau \geq s\}\} \in \mathcal{B}([0, t]) \times \mathcal{F}_{\tau \wedge t}. \quad (1.6.1)$$

It is sufficient to prove (1.6.1) for sets $C$ in a $\pi$-system generating $\mathcal{B}([0, t]) \times \mathcal{F}_s$. Let $\mathcal{I} = \{B \times F \mid B \in \mathcal{B}([0, t]), F \in \mathcal{F}_s\}$, then $\mathcal{I}$ is a $\pi$-system and $\mathcal{B}([0, t]) \times \mathcal{F}_s = \sigma(\mathcal{I})$. 
Take $B \times F \in \mathcal{L}$. Then,
\[ B \times F \cap \{ [0, t] \times \{ t \land \tau \geq s \} \} = B \times \{ F \cap \{ t \land \tau \geq s \} \}. \]

It is now sufficient to show that
\[ \{ F \cap \{ t \land \tau \geq s \} \} \in \mathcal{F}_{I \land \tau}, \quad F \in \mathcal{F}_s. \]

This follows from an easy verification argument.

To show the desired progressive measurability, we need to show that
\[ B(A) = \{ (s, \omega) \in [0, t] \times \Omega \mid X_{s \land \tau}(\omega) \in A \} \in \mathcal{B}([0, t]) \times \mathcal{F}_{I \land \tau}, \quad A \in \mathcal{E}. \]

Split $B(A)$ according to whether $s < t \land \tau$ or $s \geq t \land \tau$. Then $B(A) = B_1(A) \cup B_2(A)$, with
\[
B_1(A) = \{ (s, \omega) \in [0, t] \times \Omega \mid X_{s \land \tau}(\omega) \in A, s < t \land \tau(\omega) \},
\]
\[
B_2(A) = \{ (s, \omega) \in [0, t] \times \Omega \mid X_{s \land \tau}(\omega) \in A, s \geq t \land \tau(\omega) \}.
\]

First, we consider $B_1(A)$. Then, with $Q_t = \{ t q \mid q \in Q \cap [0, 1] \}$, $B_1(A) = \cup_{q \in Q} B^q_1(A)$, where
\[
B^q_1(A) = \{ (s, \omega) \in [0, t] \times \Omega \mid X_s(\omega) \in A, s < q \leq t \land \tau(\omega) \}
\]
\[
= \{ (s, \omega) \in [0, q] \times \Omega \mid X_s(\omega) \in A \} \cap \{ [0, t] \times \{ \omega \mid q \leq t \land \tau(\omega) \} \}.
\]

Applying (1.6.1) and using progressive measurability of $X$, yields $B^q_1(A) \in \mathcal{B}([0, t]) \times \mathcal{F}_{I \land \tau}$.

Next, we consider $B_2(A)$. We can write
\[
B_2(A) = \{ [0, t] \times \{ \omega \mid X_{I \land \tau}(\omega) \in A \} \} \cap B_3(A),
\]
where
\[
B_3(A) = \{ (s, \omega) \in [0, t] \times \Omega \mid t \land \tau(\omega) \leq s \} = \{ [0, t] \times \Omega \} \cap \{ (s, \omega) \in [0, t] \times \Omega \mid t \land \tau(\omega) > s \} \notin \mathcal{F}_{I \land \tau}.
\]

It is hence sufficient, to show that
\[ \{ (s, \omega) \in [0, t] \times \Omega \mid t \land \tau(\omega) > s \} \in \mathcal{B}([0, t]) \times \mathcal{F}_{I \land \tau}. \]

It suffices to use a similar decomposition as we did for $B_1(A)$.

The proof of (2) is an exercise. \(\square\)

In the subsequent chapters we repeatedly need the following technical lemma. It states that every stopping time is the decreasing limit of a sequence of stopping times that take only countably many values.

**Lemma 1.6.17** Let $\tau$ be a stopping time. Then there exist stopping times $\tau_n$ that only take either finitely or countably many values and such $\tau_n \downarrow \tau$.

**Proof.** Define
\[
\tau_n = \sum_{k=1}^{2^n} \frac{k}{2^n} \mathbf{1}_{\{ \tau \in ((k-1)/2^n, k/2^n] \}} + \infty \mathbf{1}_{\{ \tau > n \}},
\]

to obtain an approximating stopping time taking only finitely many values. Or
\[
\tau_n = \sum_{k=1}^{\infty} \frac{k}{2^n} \mathbf{1}_{\{ ((k-1)/2^n, k/2^n] \}} + \infty \cdot \mathbf{1}_{\{ \tau = \infty \}},
\]
for an approximating stopping time taking countable many values. Then $\tau_n$ is a stopping time and $\tau_n \downarrow \tau$ (see Exercise 1.27). \(\square\)
Optional times Similarly, for a progressively measurable process $X$, and $\tau$ an adapted optional time, it holds that $X_\tau$ is $F_\tau$-measurable. This follows directly from Lemma 1.6.7. We can further approximate $\tau$ by a non-decreasing sequence of stopping times: take the sequence from the preceding lemma. Then $\tau_n \geq \tau_{n+1} \geq \tau$ for $n \geq 1$. By virtue of the preceding lemma $\tau_n$ is an $(F_t)_t$-stopping time for all $n$ with $\lim_{n \to \infty} \tau_n = \tau$. Moreover, for all $A \in F_\tau$ it holds that $A \cap \{\tau_n = k/2^n\} \in F_{k/2^n}$, $n, k \geq 1$. Furthermore, if $\sigma$ is a $(F_t)_t$-stopping time with $\tau \leq \sigma$ and $\tau < \sigma$ on $\{\tau < \infty\}$, then $F_{\tau+} \subset F_{\sigma}$ (cf. Exercise 1.35).

Finally... Using the notion of filtrations, we can extend the definition of BM as follows.

**Definition 1.6.18** Suppose that on a probability space $(\Omega, F, P)$ we have a filtration $(F_t)_{t \geq 0}$ and an adapted stochastic process $W = (W_t)_{t \geq 0}$. Then $W$ is called a *(standard) Brownian motion* (or a Wiener process) with respect to the filtration $(F_t)_t$ if

i) $W_0 = 0$;

ii) (independence of increments) $W_t - W_s$ is independent of $F_s$ for all $s \leq t$;

iii) (stationarity of increments) $W_t - W_s \overset{d}{=} \mathcal{N}(0, t-s)$ distribution;

iv) all sample paths of $W$ are continuous.

Clearly, process $W$ that is a BM in the sense of the ‘old’ Definition 1.1.2 is a BM with respect to its natural filtration. If in the sequel we do not mention the filtration of a BM explicitly, we mean the natural filtration. However, we will see that it is sometimes necessary to consider Brownian motions with larger filtrations as well.
1.7 Exercises

Exercise 1.1 Show the claim in the proof of Theorem 1.1.6 that the system \( \mathcal{I} \) described there is a \( \pi \)-system for the \( \sigma \)-algebra \( \sigma(N_n, n \leq s) \).

Exercise 1.2 Complete the proof of Corollary 1.2.4. Give full details.

Exercise 1.3 Give an example of two processes that are versions of each other, but not modifications.

Exercise 1.4 Prove that the process \( Y \) defined in the proof of Theorem 1.3.3 is indeed a modification of the process \( X \). See remark in Step 4 of the proof of this theorem.

Exercise 1.5 An example of a right-continuous but not continuous stochastic process \( X \) is the following. Let \( \Omega = [0, 1] \), \( \mathcal{F} = \mathcal{B}([0, 1]) \) and \( \mathbb{P} = \lambda \) is the Lebesgue measure on \([0, 1]\). Let \( Y \) be the identity map on \( \Omega \), i.e. \( Y(\omega) = \omega \). Define a stochastic process \( X = (X_t)_{t \in [0, 1]} \) by \( X_t(\omega) = 1_{\{Y(\omega) \leq t\}} \). Hence, \( X_t(\omega) = 1_{\{Y(\omega) \leq t\}} = 1_{\{0, t\}}(\omega) \).

The process \( X \) does not satisfy the conditions of Kolmogorov’s Continuity Criterion, but it does satisfy the condition

\[ E|X_t - X_s|^\alpha \leq K|t - s|, \]

for any \( \alpha > 0 \) and \( K = 1 \). Show this.

Exercise 1.6 Suppose that \( X \) and \( Y \) are modifications of each other with values in a Polish space \((E, \mathcal{E})\), and for both \( X \) and \( Y \) all sample paths are either left or right continuous. Let \( T \) be an interval in \( \mathbb{R} \). Show that

\[ \mathbb{P}\{X_t = Y_t, \text{ for all } t \in T\} = 1. \]

Exercise 1.7 Prove that the process \( X \) in the proof of Corollary 1.3.4 is a BM process. To this end, you have to show that \( X \) has the correct fdd’s, and that \( X \) has a.s. continuous sample paths.

Exercise 1.8 Prove Lemma 1.4.2.

Exercise 1.9 Prove parts (i,ii,iii) of Theorem 1.4.4.

Exercise 1.10 Consider the proof of the time-inversion property of Theorem 1.4.4. Prove that

\[ \mathbb{P}\left\{ \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{q \in (0, 1/n) \cap \mathbb{Q}} \{\omega : |X_q(\omega)| < 1/m\} \right\} = \mathbb{P}\left\{ \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{q \in (0, 1/n) \cap \mathbb{Q}} \{\omega : |W_q(\omega)| < 1/m\} \right\} = 1. \]
Exercise 1.11 Let $W$ be a BM and define $X_t = W_{1-t} - W_1$ for $t \in [0, 1]$. Show that $(X_t)_{t \in [0,1]}$ is a BM as well.

Exercise 1.12 Let $W$ be a BM and fix $t > 0$. Define the process $B$ by

$$B_s = W_{s \wedge t} - (W_s - W_{s \wedge t}) = \begin{cases} W_s, & s \leq t \\ 2W_t - W_s, & s > t. \end{cases}$$

Draw a picture of the processes $W$ and $B$ and show that $B$ is again a BM. We will see another version of this so-called reflection principle in Chapter 3.

Exercise 1.13 i) Let $W$ be a BM and define the process $X_t = W_t - tW_1$, $t \in [0, 1]$. Determine the mean and covariance functions of $X$.

ii) The process $X$ of part (i) is called the (standard) Brownian bridge on $[0, 1]$, and so is every other continuous Gaussian process indexed by the interval $[0,1]$ that has the same mean and covariance function. Show that the processes $Y$ and $Z$ defined by $Y_t = (1-t)W_{1/(1-t)}$, $t \in [0,1]$, and $Y_1 = 0$ and $Z_0 = 0$, $Z_t = tW_{(1/t)-1}$, $t \in (0,1)$ are standard Brownian bridges.

Exercise 1.14 Let $H \in (0, 1)$ be given. A continuous, zero-mean Gaussian process $X$ with covariance function $2EX_sX_t = (t^{2H} + s^{2H} - |t - s|^{2H})$ is called a fractional Brownian motion (fBM) with Hurst index $H$. Show that the fBM with Hurst index 1/2 is simply the BM. Show that if $X$ is a fBM with Hurst index $H$, then for all $a > 0$ the process $a^{-H}X_{at}$ is a fBM with Hurst index $H$ as well.

Exercise 1.15 Let $W$ be a Brownian motion and fix $t > 0$. For $n \in \mathbb{Z}_+$, let $\pi_n$ be a partition of $[0, t]$ given by $0 = t^n_0 < t^n_1 < \cdots < t^n_k = t$ and suppose that the mesh $\|\pi_n\| = \max_k |t^n_k - t^n_{k-1}|$ tends to zero as $n \to \infty$. Show that

$$\sum_k (W^n_{t_k} - W^n_{t_{k-1}})^2 \overset{L^2}{\to} t,$$

as $n \to \infty$. Hint: show that the expectation of the sum tends to $t$ and the variance to 0.

Exercise 1.16 Show that if $\mathcal{F}_t$ is a filtration, then $\mathcal{F}_{t+}$ is a filtration as well.

Exercise 1.17 Prove that the collection $\mathcal{F}_\tau$ associated with a stopping time $\tau$ is a $\sigma$-algebra.

Exercise 1.18 Show that if $\sigma, \tau$ are stopping times with $\sigma \leq \tau$, then $\mathcal{F}_\sigma \subseteq \mathcal{F}_\tau$.

Exercise 1.19 Let $\sigma$ and $\tau$ be two $(\mathcal{F}_t)$-stopping times. Show that $\{\sigma \leq \tau\} \subset \mathcal{F}_\sigma \cap \mathcal{F}_\tau$.

Exercise 1.20 If $\sigma$ and $\tau$ are stopping times w.r.t. the filtration $\mathcal{F}_t$, show that $\sigma \land \tau$ and $\sigma \lor \tau$ are stopping times as well. Determine the associated $\sigma$-algebras. **Hint:** show that $A \in \mathcal{F}_{\sigma \lor \tau}$ implies $A \cap \{\sigma \leq \tau\} \in \mathcal{F}_\tau$.

Exercise 1.21 If $\sigma$ and $\tau$ are stopping times w.r.t. the filtration $\mathcal{F}_t$, show that $\sigma + \tau$ is a stopping time as well. Hint: for $t > 0$ write

$$\{\sigma + \tau > t\} = \{\tau = 0, \sigma > t\} \cup \{0 < \tau < t, \sigma + \tau > t\} \cup \{\tau > t, \sigma = 0\} \cup \{\tau \geq t, \sigma > 0\}.$$ Only for the second event on the right-hand side it is non-trivial to prove that it belongs to $\mathcal{F}_t$. Now observe that if $\tau > 0$, then $\sigma + \tau > t$ if and only if there exists a positive $q \in \mathbb{Q}$, such that $q < \tau$ and $\sigma + q > t$. 

1.7. EXERCISES
Exercise 1.22 Show that if $\sigma$ and $\tau$ are stopping times w.r.t. the filtration $(\mathcal{F}_t)$ and $X$ is an integrable random variable, then $\mathbb{1}_{\{\tau = \sigma\}}\mathbb{E}(X | \mathcal{F}_\tau) \overset{a.s.}{=} \mathbb{1}_{\{\tau = \sigma\}}\mathbb{E}(X | \mathcal{F}_\sigma)$. Hint: show that $\mathbb{1}_{\{\tau = \sigma\}}\mathbb{E}(X | \mathcal{F}_\tau) = \mathbb{1}_{\{\tau = \sigma\}}\mathbb{E}(X | \mathcal{F}_\tau \cap \mathcal{F}_\sigma)$.

Exercise 1.23 Show that if the filtration $(\mathcal{F}_t)$ is right-continuous, then $\tau$ is an $(\mathcal{F}_t)$-stopping time if and only if $\{\tau < t\} \in \mathcal{F}_t$ for all $t \in T$.

Exercise 1.24 Prove Lemma 1.6.5. Hint for (a): use BN Lemma 3.10.

Exercise 1.25 Prove Lemma 1.6.7.

Exercise 1.26 Show that the map $\omega \rightarrow (\tau(\omega) \wedge t, \omega)$ in the proof of Lemma 1.6.15 is measurable as a map from $(\Omega, \mathcal{F}_t)$ to $([0, t] \times \Omega, \mathcal{B}([0, t]) \times \mathcal{F}_t))$.

Exercise 1.27 Show that $\tau_n$ in the proof of Lemma 1.6.17 are indeed stopping times and that they converge to $\tau$.

Exercise 1.28 Translate the definitions of §1.6 to the special case that time is discrete, i.e. $T = \mathbb{Z}_+$.

Exercise 1.29 Let $W$ be a BM and let $Z = \{t \geq 0 \mid W_t = 0\}$ be its zero set. Show that with probability 1 the set $Z$ has Lebesgue measure 0, is closed and unbounded.

Exercise 1.30 We define the last exit time of $x$: 

$$L_x = \sup\{t > 0 : W_t = x\},$$

where $\sup\{0\} = 0$.

i) Show that $\tau_0$ is measurable ($\tau_0$ is defined in Definition 1.6.8).

ii) Show that $L_x$ is measurable for all $x$. Derive first that $\{L_x < t\} = \bigcap_{n > t}\{|W_s - x| > 0, t \leq s \leq n\}$.

iii) Show that $L_x = \infty$ a.s. for all $x$, by considering the set $\{\sup_{t \geq 0} W_t = \infty, \inf_{t \geq 0} W_t = -\infty\}$ as in the proof of Example 1.6.1.

iv) Show that for almost all $\omega \in \Omega$ there exists a strictly decreasing sequence $\{t_n(\omega)\}_{n}$, $\lim_n t_n(\omega) = 0$, such that $W(t_n)(\omega) = 0$ for all $n$. Hint: time-inversion + (iii). Hence $t = 0$ is a.s. an accumulation point of zeroes of $W$ and so $\tau_0 = 0$ a.s.

Exercise 1.31 Consider Brownian motion $W$ with continuous paths, defined on a probability space $(\Omega, \mathcal{F}, P)$. Let $Z(\omega) = \{t \geq 0, W_t(\omega) = 0\}$ be its zero set. In Problem 1.30 you have been asked to show that $t = 0$ is an accumulation point of $Z(\omega)$ for almost all $\omega \in \Omega$.

Let $\lambda$ denote the Lebesgue measure on $[0, \infty)$. Show (by interchanging the order of integration) that 

$$\int_{\Omega} \lambda(Z(\omega))dP(\omega) = 0,$$

and argue from this that $Z$ a.s. has Lebesgue measure 0, i.e. $\lambda(Z(\omega)) = 0$ for a.a. $\omega \in \Omega$. 

Exercise 1.32 Let $W$ be a BM (with continuous paths) with respect to its natural filtration $(\mathcal{F}_t^W)$. Define for $a > 0$
\[ S_a = \inf\{t \geq 0 : W_t > a\}. \]
i) Is $S_a$ an optional time? Justify your answer.
Let now $\sigma_a = \inf\{t \geq 0 : W_t = a\}$ be the first entrance time of $a$ and let
\[ M_a = \sup\{t \geq 0 : W_t = at\}, \]
be the last time that $W_t$ equals $at$.
ii) Is $M_a$ a stopping time? Justify your answer. Show that $M_a < \infty$ with probability 1 (you could use time-inversion for BM).
iii) Show that $M_a$ has the same distribution as $1/\sigma_a$.

Exercise 1.33 Let $X = (X_t)_{t \geq 0}$ be a Gaussian, zero-mean stochastic process starting from 0, i.e. $X_0 = 0$. Moreover, assume that the process has stationary increments, meaning that for all $t_1 \geq s_1, t_2 \geq s_2, \ldots, t_n \geq s_n$, the distribution of the vector $(X_{t_1} - X_{s_1}, \ldots, X_{t_n} - X_{s_n})$ only depends on the time points through the differences $t_1 - s_1, \ldots, t_n - s_n$.
a) Show that for all $s, t \geq 0$
\[ \mathbb{E}X_sX_t = \frac{1}{2}(v(s) + v(t) - v(|t - s|)), \]
where the function $v$ is given by $v(t) = \mathbb{E}X_t^2$.
In addition to stationarity of the increments we now assume that $X$ is $H$-self similar for some parameter $H > 0$. Recall that this means that for every $a > 0$, the process $(X_{at})_t$ has the same finite dimensional distributions as $(a^H X_t)_t$.
b) Show that the variance function $v(t) = \mathbb{E}X_t^2$ must be of the form $v(t) = Ct^{2H}$ for some constant $C \geq 0$.
In view of the (a,b) we now assume that $X$ is a zero-mean Gaussian process with covariance function
\[ \mathbb{E}X_sX_t = \frac{1}{2}(s^{2H} + t^{2H} - |t - s|^{2H}), \]
or some $H > 0$.
c) Show that we must have $H \leq 1$. (Hint: you may use that by Cauchy-Schwartz, the (semi-)metric $d(s, t) = \sqrt{\mathbb{E}(X_s - X_t)^2}$ on $[0, \infty)$ satisfies the triangle inequality).
d) Show that for $H = 1$, we have $X_t = tZ$ a.s., for a standard normal random variable $Z$ not depending on $t$.
e) Show that for every value of the parameter $H \in (0, 1]$, the process $X$ has a continuous modification.

Exercise 1.34 Let $\tau$ be an $\{\mathcal{F}_t\}_t$-optional time. Show that $\mathcal{F}_{\tau^+}$ is a $\sigma$-algebra, with respect to which $\tau$ is measurable. Show $\mathcal{F}_{\tau^+} = \{A \in \mathcal{F} : A \cap \{\tau < t\} \in \mathcal{F}_t, t \geq 0\}$. Let $\sigma, \tau$ be $\{\mathcal{F}_t\}_t$-optional times. Prove that $\mathcal{F}_{(\sigma \wedge \tau)^+} = \mathcal{F}_{\sigma^+} \cap \mathcal{F}_{\tau^+}$. If additionally $\sigma \leq \tau$ everywhere, then finally show that $\mathcal{F}_{\sigma^+} \subset \mathcal{F}_{\tau^+}$.

Exercise 1.35 Show the validity of the assertion in the paragraph on optional times below Lemma 1.6.17.
Chapter 2

Martingales

2.1 Definition and examples

In this chapter we introduce and study a very important class of stochastic processes: the so-called martingales. Martingales arise naturally in many branches of the theory of stochastic processes. In particular, they are very helpful tools in the study of BM. In this section, the index set $T$ is an arbitrary interval of $\mathbb{Z}_+$ and $\mathbb{R}_+$.

**Definition 2.1.1** An $\{(F_t)\}$-adapted, real-valued process $M$ is called a **martingale** (with respect to the filtration $\{(F_t)\}$) if

i) $\mathbb{E}|M_t| < \infty$ for all $t \in T$;

ii) $\mathbb{E}(M_t | F_s) = M_s$ for all $s \leq t$.

If property (ii) holds with ‘$\geq$’ (resp. ‘$\leq$’) instead of ‘$=$’, then $M$ is called a **submartingale** (resp. **supermartingale**).

Intuitively, a martingale is a process that is ‘constant on average’. Given all information up to time $s$, the best guess for the value of the process at time $t \geq s$ is simply the current value $M_s$.

In particular, property (ii) implies that $\mathbb{E}M_t = \mathbb{E}M_0$ for all $t \in T$. Likewise, a submartingale is a process that increases on average, and a supermartingale decreases on average. Clearly, $M$ is a submartingale if and only if $-M$ is a supermartingale and $M$ is a martingale if it is both a submartingale and a supermartingale. The basic properties of conditional expectations give us the following result and examples.

Review BN §7 Conditional expectations.

**N.B.** Let $T = \mathbb{Z}_+$. The tower property implies that (sub-, super-)martingale property (ii) is implied by (ii’) $\mathbb{E}\{M_{n+1} | F_n\} = M_{n}(\geq, \leq)$ a.s. for $n \in \mathbb{Z}_+$.

**Example 2.1.1** Let $X_n, n = 1, \ldots,$ be a sequence of i.i.d. real-valued integrable random variables. Take e.g. the filtration $F_n = \sigma(X_1, \ldots, X_n)$. Then $M_n = \sum_{k=1}^n X_k$ is a martingale if $\mathbb{E}X_1 = 0$, a submartingale if $\mathbb{E}X_1 > 0$ and a supermartingale if $\mathbb{E}X_1 < 0$. The process $M = (M_n)_n$ can be viewed as a random walk on the real line.

If $\mathbb{E}X_1 = 0$, but $X_1$ is square integrable, $M'_n = M_n^2 - n\mathbb{E}X_1^2$ is a martingale.
2.2. DISCRETE-TIME MARTINGALES

Example 2.1.2 (Doob martingale) Suppose that $X$ is an integrable random variable and $(\mathcal{F}_t)_{t \in T}$ a filtration. For $t \in T$, define $M_t = \mathbb{E}(X | \mathcal{F}_t)$, or, more precisely, let $M_t$ be a version of $\mathbb{E}(X | \mathcal{F}_t)$. Then $M = (M_t)_{t \in T}$ is an $(\mathcal{F}_t)$-martingale and $M$ is uniformly integrable (see Exercise 2.1).

Example 2.1.3 Suppose that $M$ is a martingale and that $\phi$ is a convex function such that $\mathbb{E}|\phi(M_t)| < \infty$ for $t \in T$. Then the process $\phi(M)$ is a submartingale. The same is true if $M$ is a submartingale and $\phi$ is an increasing, convex function (see Exercise 2.2).

BM generates many examples of martingales. The most important ones are given in the following example.

Example 2.1.4 Let $W$ be a BM. Then the following processes are martingales with respect to the same filtration:

i) $W$ itself;

ii) $W^2_t - t$;

iii) for every $a \in \mathbb{R}$ the process $\exp\{aW_t - a^2t/2\}$;

You are asked to prove this in Exercise 2.3.

Example 2.1.5 Let $N$ be a Poisson process with rate $\lambda$. Then $\{N(t) - \lambda t\}_t$ is a martingale.

In the next section we first develop the theory for discrete-time martingales. The generalisation to continuous time is discussed in section 2.3. In section 2.4 we continue our study of BM.

2.2 Discrete-time martingales

In this section we restrict ourselves to martingales (and filtrations) that are indexed by (a subinterval of) $\mathbb{Z}_+$. We will assume the underlying filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n}, \mathbb{P})$ to be fixed. Note that as a consequence, it only makes sense to consider $\mathbb{Z}_+$-valued stopping times. In discrete time, $\tau$ is a stopping time with respect to the filtration $(\mathcal{F}_n)_{n \in \mathbb{Z}_+}$, if $\{\tau \leq n\} \in \mathcal{F}_n$ for all $n \in \mathbb{Z}_+$.

2.2.1 Martingale transforms

If the value of a process at time $n$ is already known at time $n - 1$, we call a process predictable. The precise definition is as follows.

Definition 2.2.1 We call a discrete-time process $X$ predictable with respect to the filtration $(\mathcal{F}_n)_n$ if $X_n$ is $\mathcal{F}_{n-1}$-measurable for every $n$.

In the following definition we introduce discrete-time ‘integrals’. This is a useful tool in martingale theory.
**Definition 2.2.2** Let $M$ and $X$ be two discrete-time processes. We define the process $X \cdot M$ by $(X \cdot M)_0 = 0$ and for $n \geq 1$

$$(X \cdot M)_n = \sum_{k=1}^{n} X_k(M_k - M_{k-1}).$$

We call $X \cdot M$ the discrete integral of $X$ with respect to $M$. If $M$ is a (sub-, super-)martingale, it is often called the martingale transform of $M$ by $X$.

One can view martingale transforms as a discrete version of the Ito integral. The predictability plays a crucial role in the construction of the Ito integral.

The following lemma explains why these ‘integrals’ are so useful: the integral of a predictable process with respect to a martingale is again a martingale.

**Lemma 2.2.3** Let $X$ be a predictable process, such that for all $n$ there exists a constant $K_n$ such that $|X_1|, \ldots, |X_n| \leq K_n$. If $M$ is an martingale, then $X \cdot M$ is a martingale. If $M$ be a submartingale (resp. a supermartingale) and $X$ is non-negative then $X \cdot M$ is a submartingale (resp. supermartingale) as well.

**Proof.** Put $Y = X \cdot M$. Clearly $Y$ is adapted. Since $X$ is bounded, say $|X_n| \leq K$ a.s., for all $n$, we have $E|Y_n| \leq 2K_n \sum_{k \leq n} E|M_k| < \infty$. Now suppose first that $M$ is a submartingale and $X$ is non-negative. Then a.s.

$$E(Y_n | \mathcal{F}_{n-1}) = E(Y_{n-1} + X_n(M_n - M_{n-1}) | \mathcal{F}_{n-1}) = Y_{n-1} + X_nE(M_n - M_{n-1} | \mathcal{F}_{n-1}) \geq Y_{n-1},$$

Consequently, $Y$ is a submartingale. If $M$ is a martingale, the last inequality is an equality, irrespective of the sign of $X_n$. This implies that then $Y$ is a martingale as well. QED

Using this lemma, it is easy to see that a stopped (sub-, super-)martingale is again a (sub-, super-)martingale. However, a direct proof is too simple by itself!

**Theorem 2.2.4** Let $M$ be a $(\mathcal{F}_n)_n$ (sub-, super-)martingale and $\tau$ an $(\mathcal{F}_n)_n$-stopping time. Then the stopped process $M^\tau$ is an $(\mathcal{F}_n)_n$ and $(\mathcal{F}_{\tau \wedge n})_n$ (sub-, super-)martingale as well.

**Proof.** Define the process $X$ by $X_n = 1_{\{\tau \geq n\}}$. Verify that $M^\tau = M_0 + X \cdot M$. Since $\tau$ is a stopping time, we have that $\{\tau \geq n\} = \{\tau \leq n - 1\}^c \in \mathcal{F}_{n-1}$. Hence the process $X$ is predictable. It is also a bounded process, and so the statement follows from the preceding lemma.

We will also give a direct proof. First note that $E|M^\tau_t| = E|M_{t \wedge \tau}| \leq \sum_{n=0}^{t} E|M_n| < \infty$ for $t \in T$. Write

$$M^\tau_t = M_{t \wedge \tau} = \left( \sum_{n=0}^{t-1} \mathbf{1}_{\{\tau = n\}} + \mathbf{1}_{\{\tau \geq t\}} \right) M_{t \wedge \tau} = \sum_{n=0}^{t-1} M_n \mathbf{1}_{\{\tau = n\}} + M_t \mathbf{1}_{\{\tau \geq t\}}.$$
Taking conditional expectations yields

\[
E(M_t^\tau \mid F_{t-1}) = \sum_{n=0}^{t-1} M_n 1_{\{\tau = n\}} + 1_{\{\tau > t\}} E(M_t \mid F_{t-1}),
\]

since \(\{\tau \geq t\} \in F_{t-1}\). The rest follows immediately.

QED

The following result can be viewed as a first version of the so-called optional sampling theorem. The general version will be discussed in section 2.2.5.

**Theorem 2.2.5** Let \(M\) be a (sub)martingale and let \(\sigma, \tau\) be two stopping times such that \(\sigma \leq \tau \leq K\), for some constant \(K > 0\). Then

\[
E(M_{\tau} \mid F_{\sigma})_{(\geq)} = M_{\sigma}, \quad \text{a.s.} \tag{2.2.1}
\]

An adapted integrable process \(M\) is a martingale if and only if

\[
EM_{\tau} = EM_{\sigma},
\]

for any pairs of bounded stopping times \(\sigma \leq \tau\).

**Proof.** Suppose first that \(M\) is a martingale. Define the predictable process \(X_n = 1_{\{\tau \geq n\}} - 1_{\{\sigma \geq n\}}\). Note that \(X_n \geq 0\) a.s.! Hence, \(X \cdot M = M^\tau - M^\sigma\). By Lemma 2.2.3 the process \(X \cdot M\) is a martingale, hence \(E(M_n^\tau - M_n^\sigma) = E(X \cdot M)_n = 0\) for all \(n\). Since \(\sigma \leq \tau \leq K\) a.s., it follows that

\[
EM_{\tau} = EM_{K} = EM_{K}^\tau = EM_{\sigma}.
\]

Now we take \(A \in F_{\sigma}\) and we define the ‘truncated’ random times

\[
\sigma^A = \sigma 1_{\{A\}}, \quad \tau^A = \tau 1_{\{A\}} + K 1_{\{A^c\}}. \tag{2.2.2}
\]

By definition of \(F_{\sigma}\) it holds for every \(n\) that

\[
\{\sigma^A \leq n\} = (A \cap \{\sigma \leq n\}) \cup (A^c \cap \{K \leq n\}) \in F_{n},
\]

and so \(\sigma^A\) is a stopping time. Similarly, \(\tau^A\) is a stopping time and clearly \(\sigma^A \leq \tau^A \leq K\) a.s.

By the first part of the proof, it follows that \(EM_{\sigma A} = EM_{\tau A}\), in other words

\[
\int_A M_{\sigma} dP + \int_{A^c} M_{K} dP = \int_A M_{\tau} dP + \int_{A^c} M_{K} dP, \tag{2.2.3}
\]

by which \(\int_A M_{\sigma} dP = \int_A M_{\tau} dP\). Since \(A \in F_{\sigma}\) is arbitrary, \(E(M_{\tau} \mid F_{\sigma}) = M_{\sigma}\) a.s. (recall that \(M_{\sigma}\) is \(F_{\sigma}\)-measurable, cf. Lemma 1.6.15).

Let \(M\) be an adapted process with \(EM_{\sigma} = EM_{\tau}\) for each bounded pair \(\sigma \leq \tau\) of stopping times. Take \(\sigma = n - 1\) and \(\tau = n\) in the preceding and use truncated stopping times \(\sigma^A\) and \(\tau^A\) as in (2.2.2) for \(A \in F_{n-1}\). Then (2.2.3) for \(A \in F_{n-1}\) and stopping times \(\sigma^A\) and \(\tau^A\) implies that \(E(M_{n} \mid F_{n-1}) = M_{n-1}\) a.s. In other words, \(M\) is a martingale.

If \(M\) is a submartingale, the same reasoning applies, but with inequalities instead of equalities.

As in the previous lemma, we will also give a direct proof of (2.2.1). First note that

\[
E(M_{K} \mid F_{n}) \geq M_{n}, \quad \text{a.s.} \quad \iff \quad EM_{K} 1_{\{F\}} \geq EM_{n} 1_{\{F\}}, \quad \forall F \in F_{n}, \tag{2.2.4}
\]
(this follows from Exercise 2.7 (a)). We will first show that
\[ \mathbb{E}(M_K | \mathcal{F}_\sigma) \geq M_\sigma \text{ a.s.} \]  
(2.2.5)

Similarly to (2.2.4) it is sufficient to show that \( \mathbb{E}1_{\{F\}} M_\sigma \leq \mathbb{E}1_{\{F\}} M_K \) for all \( F \in \mathcal{F}_\sigma \). Now,
\[
\mathbb{E}1_{\{F\}} M_\sigma = \mathbb{E}1_{\{F\}} \left( \sum_{n=0}^{K} 1_{\{\sigma = n\}} + 1_{\{\sigma > K\}} \right) M_\sigma \\
= \sum_{n=0}^{K} \mathbb{E}1_{\{F \cap \{\sigma = n\}\}} M_n \\
\leq \sum_{n=0}^{K} \mathbb{E}1_{\{F \cap \{\sigma = n\}\}} M_K \\
= \mathbb{E}1_{\{F\}} \left( \sum_{n=0}^{K} 1_{\{\sigma = n\}} + 1_{\{\sigma > K\}} \right) M_K = \mathbb{E}1_{\{F\}} M_K.
\]

In the second and fourth equalities we have used that \( \mathbb{E}1_{\{\sigma > K\}} M_\sigma = \mathbb{E}1_{\{\sigma > K\}} M_K = 0 \), since \( P\{\sigma > K\} = 0 \). In the third inequality, we have used (2.2.4) and the fact that \( F \cap \{\sigma = n\} \in \mathcal{F}_n \) (why?). This shows the validity of (2.2.5).

Apply (2.2.5) to the stopped process \( M^\tau \). This yields
\[ \mathbb{E}(M_K^\tau | \mathcal{F}_\sigma) \geq M_\sigma^\tau. \]

Now, note that \( M_K^\tau = M_\tau \) a.s. and \( M_\sigma^\tau = M_\sigma \) a.s. (why?). This shows (2.2.1).  
QED

Note that we may in fact allow that \( \sigma \leq \tau \leq K \) a.s. Lateron we need \( \sigma \leq \tau \) everywhere.

2.2.2 Inequalities

Markov’s inequality implies that if \( M \) is a discrete time process, then
\[ \lambda P\{M_n \geq \lambda\} \leq E|M_n| \]
for all \( n \in \mathbb{Z}_+ \) and \( \lambda > 0 \). Doob’s classical submartingale inequality states that for submartingales we have a much stronger result.

Theorem 2.2.6 (Doob’s submartingale inequality) Let \( M \) be a submartingale. For all \( \lambda > 0 \) and \( n \in \mathbb{N} \)
\[ \lambda P\{\max_{k \leq n} M_k \geq \lambda\} \leq E M_n 1_{\{\max_{k \leq n} M_k \geq \lambda\}} \leq E|M_n|. \]

Proof. Define \( \tau = n \wedge \inf\{k | M_k \geq \lambda\} \). This is a stopping time (see Lemma 1.6.9) with \( \tau \leq n \). By Theorem 2.2.5, we have \( E M_n \geq E M_\tau \). It follows that
\[
E M_n \geq E M_\tau 1_{\{\max_{k \leq n} M_k \geq \lambda\}} + E M_\tau 1_{\{\max_{k \leq n} M_k < \lambda\}} \\
\geq \lambda P\{\max_{k \leq n} M_k \geq \lambda\} + E M_n 1_{\{\max_{k \leq n} M_k < \lambda\}}.
\]

This yields the first inequality. The second one is obvious.  
QED
Theorem 2.2.7 (Doob’s $L^p$ inequality) If $M$ is a martingale or a non-negative submartingale and $p > 1$, then for all $n \in \mathbb{N}$

$$
E \left( \max_{k \leq n} |M_n|^p \right) \leq \left( \frac{p}{1 - p} \right)^p E |M_n|^p,
$$

provided $M$ is in $L^p$.

Proof. Define $M^* = \max_{k \leq n} |M_k|$. Assume that $M$ is defined on an underlying probability space $(\Omega, \mathcal{F}, P)$. We have for any $m \in \mathbb{N}$

$$
E(M^* \wedge m)^p = \int_{\Omega} (M^*(\omega) \wedge m)^p dP(\omega)
$$

$$
= \int_{\Omega} (M^*(\omega))^{p-1} \mathbb{1}_{M^*(\omega) \geq x} dx P(\omega)
$$

$$
= \int_{0}^{m} px^{p-1} \mathbb{1}_{M^*(\omega) \geq x} dx P(\omega)
$$

$$
= \int_{0}^{m} px^{p-1} P\{M^* \geq x\} dx,
$$

where we have used Fubini’s theorem in the last equality (non-negative integrand!). By conditional Jensen’s inequality, $|M|$ is a submartingale, and so we can apply Doob’s submartingale inequality to estimate $P\{M^* \geq x\}$. Thus

$$
P\{M^* \geq x\} \leq \frac{E(|M_n| \mathbb{1}_{M^* \geq x})}{x}.
$$

Insert this in (2.2.6), then

$$
E(M^* \wedge m)^p \leq \int_{0}^{m} px^{p-2} E(|M_n| \mathbb{1}_{M^* \geq x}) dx
$$

$$
= \int_{0}^{m} px^{p-2} \int_{\omega: M^*(\omega) \geq x} |M_n(\omega)| dP(\omega) dx
$$

$$
= \int_{\omega} |M_n(\omega)| \int_{0}^{M^*(\omega) \wedge m} x^{p-2} dx dP(\omega)
$$

$$
= \frac{p}{p-1} E(|M_n|)(M^* \wedge m)^{p-1}).
$$

By Hölder’s inequality, it follows that with $p^{-1} + q^{-1} = 1$

$$
E|M^* \wedge m|^p \leq \frac{p}{p-1} (E|M_n|^p)^{1/p} (E|M^* \wedge m|^{(p-1)/q})^{1/q}.
$$

Since $p > 1$ we have $q = p/(p-1)$, so that

$$
E|M^* \wedge m|^p \leq \frac{p}{p-1} (E|M_n|^p)^{1/p} (E|M^* \wedge m|^p)^{(p-1)/p}.
$$

Now take $p$th power of both sides and cancel common factors. Then

$$
E|M^* \wedge m|^p \leq \left( \frac{p}{p-1} \right)^p E|M_n|^p.
$$

The proof is completed by letting $m$ tend to infinity. QED
2.2.3 Doob decomposition

An adapted, integrable process $X$ can always be written as a sum of a martingale and a predictable process. This is called the Doob decomposition of the process $X$.

**Theorem 2.2.8** Let $X$ be an adapted, integrable process. There exists a martingale $M$ and a predictable process $A$, such that $A_0 = M_0 = 0$ and $X = X_0 + M + A$. The processes $M$ and $A$ are a.s. unique. The process $X$ is a submartingale if and only if $A$ is a.s. increasing (i.e. $P(A_n \leq A_{n+1}) = 1$).

**Proof.** Suppose first that there exist a martingale $M$ and a predictable process $A$ such that $A_0 = M_0 = 0$ and $X = X_0 + M + A$. The martingale property of $M$ and predictability of $A$ show that a.s.

$$E(X_n - X_{n-1} \mid \mathcal{F}_{n-1}) = A_n - A_{n-1} \quad \text{a.s.} \quad (2.2.7)$$

Since $A_0 = 0$ it follows that

$$A_n = \sum_{k=1}^{n} E(X_k - X_{k-1} \mid \mathcal{F}_{k-1}), \quad (2.2.8)$$

for $n \geq 1$ and hence $M_n = X_n - A_n - X_0$. This shows that $M$ and $A$ are a.s. unique.

Conversely, given a process $X$, $(2.2.8)$ defines a predictable process $A$. It is easily seen that the process $M$ defined by $M = X - A - X_0$ is a martingale. This proves the existence of the decomposition.

Equation $(2.2.7)$ shows that $X$ is a submartingale if and only if $A$ is increasing. QED

An important application of the Doob decomposition is the following.

**Corollary 2.2.9** Let $M$ be a martingale with $EM_n^2 < \infty$ for all $n$. Then there exists an a.s. unique predictable, increasing process $A$ with $A_0 = 0$ such that $M^2 - A$ is a martingale. Moreover the random variable $A_{n+1} - A_n$ is a version of the conditional variance of $M_n$ given $\mathcal{F}_{n-1}$, i.e.

$$A_n - A_{n-1} = E\left((M_n - E(M_n \mid \mathcal{F}_{n-1}))^2 \mid \mathcal{F}_{n-1}\right) = E\left((M_n - M_{n-1})^2 \mid \mathcal{F}_{n-1}\right) \quad \text{a.s.}$$

It follows that Pythagoras’ theorem holds for square integrable martingales

$$EM_n^2 = EM_0^2 + \sum_{k=1}^{n} E(M_k - M_{k-1})^2.$$

The process $A$ is called the predictable quadratic variation process of $M$ and is often denoted by $\langle M \rangle$.

**Proof.** By conditional Jensen, it follows that $M^2$ is a submartingale. Hence Theorem 2.2.8 applies. The only thing left to prove is the statement about conditional variance. Since $M$ is a martingale, we have a.s.

$$E((M_n - M_{n-1})^2 \mid \mathcal{F}_{n-1}) = E(M_n^2 - 2M_nM_{n-1} + M_{n-1}^2 \mid \mathcal{F}_{n-1})$$

$$= E(M_n^2 \mid \mathcal{F}_{n-1}) - 2M_nE(M_n \mid \mathcal{F}_{n-1}) + M_{n-1}^2$$

$$= E(M_n^2 \mid \mathcal{F}_{n-1}) - M_{n-1}^2$$

$$= E(M_n^2 - M_{n-1}^2 \mid \mathcal{F}_{n-1}) = A_n - A_{n-1}. \quad \text{QED}$$
2.2. DISCRETE-TIME MARTINGALES

Using the Doob decomposition in combination with the submartingale inequality yields the following result.

**Theorem 2.2.10** Let $X$ be a sub- or supermartingale. For all $\lambda > 0$ and $n \in \mathbb{Z}_+$

$$\lambda P\{\max_{k \leq n} |X_k| \geq 3\lambda\} \leq 4E|X_0| + 3E|X_n|.$$ 

**Proof.** Suppose that $X$ is a submartingale. By the Doob decomposition theorem there exist a martingale $M$ and an increasing, predictable process $A$ such that $M_0 = A_0 = 0$ and $X = X_0 + M + A$. By the triangle inequality and the fact that $A$ is increasing

$$P\{\max_{k \leq n} |X_k| \geq 3\lambda\} \leq P\{|X_0| \geq \lambda\} + P\{\max_{k \leq n} |M_k| \geq \lambda\} + P\{|A_n| \geq \lambda\}.$$ 

Hence, by Markov’s inequality and the submartingale inequality ($|M_n|$ is a submartingale!)

$$\lambda P\{\max_{k \leq n} |X_k| \geq 3\lambda\} \leq E|X_0| + E|M_n| + EA_n.$$ 

Since $M_n = X_n - X_0 - A_n$, the right-hand side is bounded by $2E|X_0| + E|X_n| + 2EA_n$. We know that $A_n$ is given by (2.2.7). Taking expectations in the latter expression shows that $EA_n = EX_n - EX_0 \leq E|X_n| + E|X_0|$. This completes the proof. QED

2.2.4 Convergence theorems

Let $M$ be a supermartingale and consider a compact interval $[a, b] \subset \mathbb{R}$. The number of upcrossings of $[a, b]$ that the process makes upto time $n$ is the number of time that the process passes from a level below $a$ to a level above $b$. The precise definition is as follows.

**Definition 2.2.11** The number $U_n[a, b]$ is the largest value $k \in \mathbb{Z}_+$, such that there exist $0 \leq s_1 < t_1 < s_2 < \cdots < s_k < t_k \leq n$ with $M_{s_i} < a$ and $M_{t_i} > b$, $i = 1, \ldots, k$.

First we define the “limit $\sigma$-algebra

$$\mathcal{F}_\infty = \sigma\left(\bigcup_n \mathcal{F}_n\right).$$

**Lemma 2.2.12** (Doob’s upcrossing lemma) Let $M$ be a supermartingale. Then for all $a < b$, the number of upcrossings $U_n[a, b]$ of the interval $[a, b]$ by $M$ upto time $n$ is an $\mathcal{F}_n$-measurable random variable and satisfies

$$(b - a)E U_n[a, b] \leq E(M_n - a)^+.$$ 

The total number of upcrossings $U_\infty[a, b]$ is $\mathcal{F}_\infty$-measurable.

**Proof.** Check yourself that $U_n[a, b]$ is $\mathcal{F}_n$-measurable and that $U_\infty[a, b]$ is $\mathcal{F}_\infty$-measurable. Consider the bounded, predictable process $X$ given by $X_0 = 1_{\{M_0 < a\}}$ and

$$X_n = 1_{\{X_{n-1} = 1\}}1_{\{M_{n-1} \leq b\}} + 1_{\{X_{n-1} = 0\}}1_{\{M_{n-1} < a\}}, \quad n \in \mathbb{Z}_+.$$
Define \( Y = X \cdot M \). The process \( X \) equals 0, until \( M \) drops below level \( a \), then stays until \( M \) gets above \( b \) etc. So every completed upcrossing of \([a, b]\) increases the value of \( Y \) by at least \( b - a \). If the last upcrossing has not yet been completed at time \( n \), then this may reduce \( Y \) by at most \((M_n - a)^-\). Hence

\[
Y_n \geq (b - a)U_n[a, b] - (M_n - a)^-.
\]

By Lemma 2.2.3, the process \( Y = X \cdot M \) is a supermartingale. In particular \( \mathbb{E}Y_n \leq \mathbb{E}Y_0 = 0 \). The proof is completed by taking expectations in both sides of (2.3.1).

Observe that the upcrossing lemma implies for a supermartingale \( M \) that is bounded in \( L^1 \) (i.e. \( \sup_n \mathbb{E}|M_n| < \infty \)) that \( \mathbb{E}U_\infty[a, b] < \infty \) for all \( a \leq b \). In particular, the total number \( U_\infty[a, b] \) of upcrossings of the interval \([a, b]\) is almost surely finite. The proof of the classical martingale convergence theorem is now straightforward.

**Theorem 2.2.13 (Doob’s martingale convergence theorem)** If \( M \) is a supermartingale that is bounded in \( L^1 \), then \( M_n \) converges a.s. to a finite \( \mathcal{F}_\infty \)-measurable limit \( M_\infty \) as \( n \to \infty \), with \( \mathbb{E}|M_\infty| < \infty \).

**Proof.** Assume that \( M \) is defined on the underlying probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Suppose that \( M(\omega) \) does not converge to a limit in \([-\infty, \infty]\). Then there exist two rationals \( a < b \) such that \( \liminf M_n(\omega) < a < b < \limsup M_n(\omega) \). In particular, we must have \( U_\infty[a, b](\omega) = \infty \). By Doob’s upcrossing lemma \( \mathbb{P}\{U_\infty[a, b] = \infty\} = 0 \). Now note that

\[
A := \{\omega \mid M(\omega) \text{ does not converge to a limit in } [-\infty, \infty]\} \subset \bigcup_{a,b \in \mathbb{Q} : a < b} \{\omega \mid U_\infty[a, b](\omega) = \infty\}.
\]

Hence \( \mathbb{P}(A) \leq \sum_{a,b \in \mathbb{Q} : a < b} \mathbb{P}\{U_\infty[a, b] = \infty\} = 0 \). This implies that \( M_n \) a.s. converges to a limit \( M_\infty \) in \([-\infty, \infty]\). Moreover, in view of Fatou’s lemma

\[
\mathbb{E}|M_\infty| = \mathbb{E}(\liminf |M_n|) \leq \liminf \mathbb{E}|M_n| \leq \sup \mathbb{E}|M_n| < \infty.
\]

It follows that \( M_\infty \) is a.s. finite and it is integrable. Note that \( M_n \) is \( \mathcal{F}_n \)-measurable, hence it is \( \mathcal{F}_\infty \)-measurable. Since \( M_\infty = \lim_{n \to \infty} M_n \) is the limit of \( \mathcal{F}_\infty \)-measurable maps, it is \( \mathcal{F}_\infty \)-measurable as well (see MTP-lecture notes).

If the supermartingale \( M \) is not only bounded in \( L^1 \) but also uniformly integrable, the in addition to a.s. convergence we have convergence in \( L^1 \). Moreover, in the case, the whole sequence \( M_1, \ldots, M_n \) is a supermartingale.

**Theorem 2.2.14** Let \( M \) be a supermartingale that is bounded in \( L^1 \). Then \( M_n \overset{L^1}{\to} M_\infty \), \( n \to \infty \), if and only if \( \{M_n : n \in \mathbb{Z}_+\} \) is uniformly integrable, where \( M_\infty \) is integrable and \( \mathcal{F}_\infty \)-measurable. In that case

\[
\mathbb{E}(M_\infty | \mathcal{F}_n) \leq M_n, \quad \text{a.s.}
\]

(2.2.10)

If in addition \( M \) is a martingale, then there is equality in (2.2.10), in other words, \( M \) is a Doob martingale.
Proof. By virtue of Theorem 2.2.13 $M_n \to M_\infty$ a.s., for a finite random variable $M_\infty$. BN Theorem 8.5 implies the first statement. To prove the second statement, suppose that $M_n \overset{L^1}{\to} M_\infty$. Since $M$ is a supermartingale, we have

$$\mathbb{E}1_{\{A\}}M_m \leq \mathbb{E}1_{\{A\}}M_n, \quad A \in \mathcal{F}_n, \ m \geq n. \quad (2.2.11)$$

Since $|1_{\{A\}}M_m - 1_{\{A\}}M_\infty| \leq 1_{\{A\}}|M_m - M_\infty| \leq |M_m - M_\infty|$, it follows directly that $1_{\{A\}}M_m \overset{L^1}{\to} 1_{\{A\}}M_\infty$. Taking the limit $m \to \infty$ in (2.2.11) yields

$$\mathbb{E}1_{\{A\}}M_\infty \leq \mathbb{E}1_{\{A\}}M_n, \quad A \in \mathcal{F}_n.$$  

This implies (see BN Exercise 2.7(a)) that $\mathbb{E}(M_\infty | \mathcal{F}_n) \leq M_n$ a.s. QED

Hence uniformly integrable martingales that are bounded in $L^1$, are Doob martingales. On the other hand, let $X$ be an $\mathcal{F}$-measurable, integrable random variable and let $(\mathcal{F}_n)_n$ be a filtration. Then (Example 2.1.2) $\mathbb{E}(X | \mathcal{F}_n)$ is a uniformly integrable Doob martingale. By uniform integrability, it is bounded in $L^1$. For Doob martingales, we can identify the limit explicitly in terms of the limit $\sigma$-algebra $\mathcal{F}_\infty$.

**Theorem 2.2.15 (Lévy’s upward theorem)** Let $X$ be an integrable random variable, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $(\mathcal{F}_n)_n$ be a filtration, $\mathcal{F}_n \subset \mathcal{F}$, for all $n$. Then as $n \to \infty$

$$\mathbb{E}(X | \mathcal{F}_n) \to \mathbb{E}(X | \mathcal{F}_\infty),$$

a.s. and in $L^1$.

Proof. The process $M_n = \mathbb{E}(X | \mathcal{F}_n)$ is uniformly integrable (see Example 2.1.2), hence bounded in $L^1$ (explain!). By Theorem 2.2.14 $M_n \to M_\infty$ a.s. and in $L^1$, as $n \to \infty$ with $M_\infty$ integrable and $\mathcal{F}_\infty$-measurable. It remains to show that $M_\infty = \mathbb{E}(X | \mathcal{F}_\infty)$ a.s. Note that

$$\mathbb{E}1_{\{A\}}M_\infty = \mathbb{E}1_{\{A\}}M_n = \mathbb{E}1_{\{A\}}X, \quad A \in \mathcal{F}_n, \quad (2.2.12)$$

where we have used Theorem 2.2.14 for the first equality and the definition of $M_n$ for the second. First assume that $X \geq 0$, then $M_n = \mathbb{E}(X | \mathcal{F}_n) \geq 0$ a.s. (see BN Lemma 7.2 (iv)), hence $M_\infty \geq 0$ a.s.

As in the construction of the conditional expectation we will associate measures with $X$ and $M_\infty$ and show that they agree on a $\pi$-system for $\mathcal{F}_\infty$. Define measures $Q_1$ and $Q_2$ on $(\Omega, \mathcal{F}_\infty)$ by

$$Q_1(A) = \mathbb{E}1_{\{A\}}X, \quad Q_2(A) = \mathbb{E}1_{\{A\}}M_\infty.$$  

(Check that these are indeed measures). By virtue of (2.2.12) $Q_1$ and $Q_2$ agree on the $\pi$-system (algebra) $\cup_\mathcal{F}_n$. Moreover, $Q_1(\Omega) = Q_2(\Omega) = \mathbb{E}X$ since $\Omega \in \mathcal{F}_n$. By virtue of BN Lemma 1.1, $Q_1$ and $Q_2$ agree on $\sigma(\cup_\mathcal{F}_n)$. This implies by definition of conditional expectation that $M_\infty = \mathbb{E}(X | \mathcal{F}_\infty)$ a.s.

Finally we consider the case of general $\mathcal{F}$-measurable $X$. Then $X = X^+ - X^-$, is the difference of two non-negative $\mathcal{F}$-measurable functions $X^+$ and $X^-$. Use the linearity of conditional expectation. QED
The message here is that one cannot know more than what one can observe. We will also need the corresponding result for decreasing families of $\sigma$-algebras. If we have a filtration of the form $(\mathcal{F}_n)_{n \in \mathbb{Z}_+}$, i.e., a collection of $\sigma$-algebras such that $\mathcal{F}_{-(n+1)} \subseteq \mathcal{F}_n$, then we define

$$\mathcal{F}_{-\infty} = \bigcap_n \mathcal{F}_n.$$ 

**Theorem 2.2.16 (Lévy-Doob downward theorem)** Let $(\mathcal{F}_n \mid n \in \mathbb{Z}_+)$ be a collection of $\sigma$-algebras, such that $\mathcal{F}_{-(n+1)} \subseteq \mathcal{F}_n$ for every $n$, and let $M = (\cdots, M_2, M_1)$ be a supermartingale, i.e.

$$\mathbb{E}(M_m \mid \mathcal{F}_n) \leq M_n \quad {\text{a.s.}} \quad {\text{for all}} \quad n \leq m \leq -1.$$ 

If $\sup \mathbb{E}M_n < \infty$, then the process $M$ is uniformly integrable and the limit

$$M_{-\infty} = \lim_{n \to \infty} M_n$$

exists a.s. and in $L^1$. Moreover,

$$\mathbb{E}(M_{-n} \mid \mathcal{F}_{-\infty}) \leq M_{-\infty} \quad {\text{a.s.}}$$

(2.2.13)

If $M$ is a martingale, we have equality in (2.2.13) and in particular $M_{-\infty} = \mathbb{E}(M_{-1} \mid \mathcal{F}_{-\infty})$.

**Proof.** For every $n \in \mathbb{Z}_+$ the upcrossing inequality applied to the supermartingale

$$(M_{-n}, M_{-(n-1)}, \ldots, M_1)$$

yields $(b - a)\mathbb{E}U_n[a, b] \leq \mathbb{E}(M_{-1} - a)^-$ for every $a < b$. By a similar reasoning as in the proof of Theorem 2.2.13, we see that the limit $M_{-\infty} = \lim_{n \to -\infty} M_n$ exists and is finite almost surely.

Next, we would like to show uniform integrability. For all $K > 0$ and $n \in \mathbb{Z}_+$ we have

$$\int_{\{|M_{-n}| > K\}} |M_{-n}|dP = \mathbb{E}M_{-n} - \int_{M_{-n} \leq K} M_{-n}dP - \int_{M_{-n} < -K} M_{-n}dP.$$ 

The sequence $\mathbb{E}M_{-n}$ is non-decreasing in $n \to -\infty$, and bounded. Hence the limit $\lim_{n \to -\infty} \mathbb{E}M_{-n}$ exists (as a finite number). For arbitrary $\epsilon > 0$, there exists $m \in \mathbb{Z}_+$, such that $\mathbb{E}M_{-n} \leq \mathbb{E}M_{-m} + \epsilon, n \geq m$. Together with the supermartingale property this implies for all $n \geq m$

$$\int_{\{|M_{-n}| > K\}} |M_{-n}|dP \leq \mathbb{E}M_{-m} + \epsilon - \int_{M_{-n} \leq K} M_{-m}dP - \int_{M_{-n} < -K} M_{-m}dP \leq \int_{\{|M_{-n}| > K\}} |M_{-m}|dP + \epsilon.$$ 

Hence to prove uniform integrability, in view of BN Lemma 8.1 it is sufficient to show that we can make $\mathbb{P}\{|M_{-n}| > K\}$ arbitrarily small for all $n$ simultaneously. By Chebychev’s inequality, it suffices to show that $\sup_n \mathbb{E}|M_{-n}| < \infty$.

To this end, consider the process $M^- = \max\{-M, 0\}$. With $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = \max\{x, 0\}$, one has $M^- = g(-M)$. The function $g$ is a non-decreasing, convex function.
Since $-M$ is a submartingale, it follows that $M^{-}$ is a submartingale (see Example 2.1.3). In particular, $EM_{n}^{-} \leq EM_{-1}^{-}$ for all $n \in \mathbb{Z}_{+}$. It follows that

$$E|M_{n}^{-}| = EM_{n}^{-} + 2EM_{-n}^{-} \leq \sup_{n} EM_{n}^{-} + 2E|M_{-1}^{-}|.$$  

Consequently

$$P(|M_{n}^{-}| > K) \leq \frac{1}{K}(\sup_{n} EM_{n}^{-} + 2E|M_{-1}^{-}|).$$

Indeed, $M$ is uniformly integrable. The limit $M_{-\infty}$ therefore exists in $L^{1}$ as well.

Suppose that $M$ is a martingale. Then $M_{-n} = E(M_{-1} | F_{-n})$ a.s. The rest follows in a similar manner as the proof of the Lévy upward theorem. QED

Note that the downward theorem includes the “downward version” of Theorem 2.2.15 as a special case. Indeed, if $X$ is an integrable random variable and $\mathcal{F}_{1} \supseteq \mathcal{F}_{2} \supseteq \cdots \supseteq \cap_{n} \mathcal{F}_{n} = \mathcal{F}_{\infty}$ is a decreasing sequence of $\sigma$-algebras, then

$$E(X | \mathcal{F}_{n}) \to E(X | \mathcal{F}_{\infty}), \quad n \to \infty$$

a.s. and in $L^{1}$. This is generalised in the following corollary to Theorems 2.2.15 and 2.2.16. It will be useful in the sequel.

**Corollary 2.2.17** Suppose that $X_{n} \to X$ a.s., and that $|X_{n}| \leq Y$ a.s. for all $n$, where $Y$ is an integrable random variable. Moreover, suppose that $\mathcal{F}_{1} \supseteq \mathcal{F}_{2} \supseteq \cdots$ (resp. $\mathcal{F}_{1} \supseteq \mathcal{F}_{2} \supseteq \cdots$) is an increasing (resp. decreasing) sequence of $\sigma$-algebras. Then $E(X_{n} | \mathcal{F}_{n}) \to E(X | \mathcal{F}_{\infty})$ a.s., where $\mathcal{F}_{\infty} = \sigma(\cup_{n} \mathcal{F}_{n})$ (resp. $\mathcal{F}_{\infty} = \cap_{n} \mathcal{F}_{n}$).

In case of an increasing sequence of $\sigma$-algebras, the corollary is known as Hunt’s lemma.

**Proof.** For $m \in \mathbb{Z}_{+}$, put $U_{m} = \inf_{n \geq m} X_{n}$ and $V_{m} = \sup_{n \geq m} X_{n}$. Since $X_{m} \to X$ a.s., necessarily $V_{m} - U_{m} \to 0$ a.s., as $m \to \infty$. Furthermore $|V_{m} - U_{m}| \leq 2Y$. Dominated convergence then implies that $E(V_{m} - U_{m}) \to 0$, as $m \to \infty$. Fix $\epsilon > 0$ and choose $m$ so large that $E(V_{m} - U_{m}) < \epsilon$. For $n \geq m$ we have

$$U_{m} \leq X_{n} \leq V_{m} \quad \text{a.s.} \tag{2.2.14}$$

Consequently $E(U_{m} | \mathcal{F}_{n}) \leq E(X_{n} | \mathcal{F}_{n}) \leq E(V_{m} | \mathcal{F}_{n})$ a.s. The processes on the left and right are martingales that satisfy the conditions of the upward (resp. downward) theorem. Letting $n$ tend to $\infty$ we obtain

$$E(U_{m} | \mathcal{F}_{\infty}) \leq \lim inf E(X_{n} | \mathcal{F}_{n}) \leq \lim sup E(X_{n} | \mathcal{F}_{n}) \leq E(V_{m} | \mathcal{F}_{\infty}) \quad \text{a.s.} \tag{2.2.15}$$

It follows that

$$0 \leq E\left(\lim sup E(X_{n} | \mathcal{F}_{n}) - \lim inf E(X_{n} | \mathcal{F}_{n})\right) \leq E\left(E(V_{m} | \mathcal{F}_{\infty}) - E(U_{m} | \mathcal{F}_{\infty})\right) \leq E(V_{m} - U_{m}) < \epsilon.$$  

Letting $\epsilon \downarrow 0$ yields that $\lim sup E(X_{n} | \mathcal{F}_{n}) = \lim inf E(X_{n} | \mathcal{F}_{n})$ a.s. and so $E(X_{n} | \mathcal{F}_{n})$ converges a.s. We wish to identify the limit. Let $n \to \infty$ in (2.2.14). Then $U_{m} \leq X \leq V_{m}$ a.s. Hence

$$E(U_{m} | \mathcal{F}_{\infty}) \leq E(X | \mathcal{F}_{\infty}) \leq E(V_{m} | \mathcal{F}_{\infty}) \quad \text{a.s.} \tag{2.2.16}$$
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Equations (2.2.15) and (2.2.16) imply that both \( \lim \mathbb{E}(X_n | \mathcal{F}_n) \) and \( \mathbb{E}(X | \mathcal{F}_\infty) \) are a.s. between \( V_m \) and \( U_m \). Consequently

\[
\mathbb{E} | \lim \mathbb{E}(X_n | \mathcal{F}_n) - \mathbb{E}(X | \mathcal{F}_\infty) | \leq \mathbb{E}(V_m - U_m) < \epsilon.
\]

By letting \( \epsilon \downarrow 0 \) we obtain that \( \lim_n \mathbb{E}(X_n | \mathcal{F}_n) = \mathbb{E}(X | \mathcal{F}_\infty) \) a.s. QED

2.2.5 Optional sampling theorems

Theorem 2.2.5 implies for a martingale \( M \) and two bounded stopping times \( \sigma \leq \tau \) that \( \mathbb{E}(M_\tau | \mathcal{F}_\sigma) = M_\sigma \). The following theorem extends this result.

**Theorem 2.2.18 (Optional sampling theorem)** Let \( M \) be a uniformly integrable (super)martingale. Then the family of random variables \( \{M_\tau | \tau \text{ is a finite stopping time} \} \) is uniformly integrable and for all stopping times \( \sigma \leq \tau \) we have

\[
\mathbb{E}(M_\tau | \mathcal{F}_\sigma)(\leq) M_\sigma \text{ a.s.}
\]

**Proof.** We will only prove the martingale statement. For the proof in case of a supermartingale see Exercise 2.14.

By Theorem 2.2.14, \( M_\infty = \lim_{n \to \infty} M_n \) exists a.s. and in \( L^1 \) and \( \mathbb{E}(M_\infty | \mathcal{F}_n) = M_n \) a.s. Now let \( \tau \) be an arbitrary stopping time and \( n \in \mathbb{Z}_+ \). Since \( \tau \land n \leq \tau \), \( \mathcal{F}_\tau \land n \subseteq \mathcal{F}_n \). By the tower property, it follows for every \( n \) that

\[
\mathbb{E}(M_\infty | \mathcal{F}_\tau \land n) = \mathbb{E}(\mathbb{E}(M_\infty | \mathcal{F}_n) | \mathcal{F}_\tau \land n) = \mathbb{E}(M_n | \mathcal{F}_\tau \land n) \text{ a.s.}
\]

By Theorem 2.2.5 we a.s. have

\[
\mathbb{E}(M_\infty | \mathcal{F}_\tau \land n) = M_\tau \land n.
\]

Now let \( n \) tend to infinity. Then the right-hand side converges a.s. to \( M_\tau \). By the Levy upward convergence theorem, the left-hand side converges a.s. and in \( L^1 \) to \( \mathbb{E}(M_\infty | \mathcal{G}) \), where

\[
\mathcal{G} = \sigma(\bigcup_n \mathcal{F}_\tau \land n).
\]

Therefore

\[
\mathbb{E}(M_\infty | \mathcal{G}) = M_\tau \text{ a.s.} \quad (2.2.17)
\]

We have to show that \( \mathcal{G} \) can be replaced by \( \mathcal{F}_\tau \). Take \( A \in \mathcal{F}_\tau \). Then

\[
\mathbb{E} 1_{\{A\}} M_\infty = \mathbb{E} 1_{\{A \land \{\tau < \infty\}\}} M_\infty + \mathbb{E} 1_{\{A \land \{\tau = \infty\}\}} M_\infty.
\]

By virtue of Exercise 2.6, relation (2.2.17) implies that

\[
\mathbb{E} 1_{\{A \land \{\tau < \infty\}\}} M_\infty = \mathbb{E} 1_{\{A \land \{\tau < \infty\}\}} M_\tau.
\]

Trivially

\[
\mathbb{E} 1_{\{A \land \{\tau = \infty\}\}} M_\infty = \mathbb{E} 1_{\{A \land \{\tau = \infty\}\}} M_\tau.
\]
Combination yields

\[ E_1(A) M_{\infty} = E_1(A) M_{\tau}, \quad A \in \mathcal{F}_\tau. \]

We conclude that \( E(M_{\infty} | \mathcal{F}_\tau) = M_{\tau} \) a.s. The first statement of the theorem follows from BN Lemma 8.4. The second statement follows from the tower property and the fact that \( \mathcal{F}_\sigma \subseteq \mathcal{F}_\tau. \)

QED

For the equality \( E(M_{\tau} | \mathcal{F}_\sigma) = M_\sigma \) a.s. in the preceding theorem to hold, it is necessary that \( M \) is uniformly integrable. There exist (positive) martingales that are bounded in \( L^1 \) but not uniformly integrable, for which the equality fails in general (see Exercise 2.27)! For nonnegative supermartingales without additional integrability properties we only have an inequality.

**Theorem 2.2.19** Let \( M \) be a nonnegative supermartingale and let \( \sigma \leq \tau \) be stopping times. Then

\[ E(M_{\tau} | \mathcal{F}_\sigma) \leq M_\sigma \quad \text{a.s.} \]

**Proof.** First note that \( M \) is bounded in \( L^1 \) and so it converges a.s. Fix \( n \in \mathbb{Z}_+ \). The stopped supermartingale \( M^{\tau \wedge n} \) is a supermartingale again (cf. Theorem 2.2.4). Check that it is uniformly integrable. Precisely as in the proof of the preceding theorem we find that

\[ E(M_{\tau \wedge n} | \mathcal{F}_\sigma) = E(M_{\infty} | \mathcal{F}_\sigma) \leq M_{\sigma \wedge n} = M_\sigma \quad \text{a.s.} \]

Since the limit exists, we have \( M_{\tau} 1_{\{\tau = \infty\}} = M_{\infty} 1_{\{\tau = \infty\}} \). By conditional Fatou

\[
E(M_{\tau} | \mathcal{F}_\sigma) \leq E(\lim \inf M_{\tau \wedge n} | \mathcal{F}_\sigma) \\
\leq \lim \inf E(M_{\tau \wedge n} | \mathcal{F}_\sigma) \\
\leq \lim \inf M_{\sigma \wedge n} = M_\sigma, \quad \text{a.s.}
\]

This proves the result. 

QED

**2.2.6 Law of Large numbers**

We have already pointed out that martingales are a generalisation of sums of i.i.d. random variables with zero expectation. For such sums, we can derive the Law of Large Numbers, Central Limit Theorem, and law of Iterated Logarithm. The question is then: do these Laws also apply to martingales? If yes, what sort of conditions do we need to require.

Here we discuss a simplest version. To this end we need some preparations.

**Theorem 2.2.20** Let \( S = (S_n = \sum_{k=1}^n X_k)_{n=1,2,...} \) be a martingale with respect to the filtration \( (\mathcal{F}_n)_{n=1,2,...} \). Assume that \( ES_n = 0 \) and that \( ES_n^2 < \infty \) for all \( n \). Then \( S_n \) converges a.s. on the set \( \{ \sum_{k=1}^\infty E(X_k^2 | \mathcal{F}_{k-1}) < \infty \} \).

**Proof.** Let \( \mathcal{F}_0 \) be the trivial \( \sigma \)-algebra. Fix \( K > 0 \) and let \( \tau = \min\{n \mid \sum_{k=1}^{n+1} E(X_k^2 | \mathcal{F}_{k-1}) > K\} \), if such \( n \) exists. Otherwise let \( \tau = \infty \). Clearly \( \tau \) is a stopping time (check yourself). Then \( S^\tau \) is a martingale.
Note that \( S_n^\tau = S_{\tau \wedge n} = \sum_{k=1}^{n} 1_{\{\tau \geq k\}} X_k \). Using the martingale property and the fact that \( \{\tau \geq k\} \in \mathcal{F}_{k-1} \), we obtain

\[
E[S_{\tau \wedge n}^2] = E\left( \sum_{k=1}^{n} 1_{\{\tau \geq k\}} X_k^2 \right) \\
= E\left( \sum_{k=1}^{n} E(1_{\{\tau \geq k\}} X_k^2 \mid \mathcal{F}_{k-1}) \right) \\
= E\left( \sum_{k=1}^{n} 1_{\{\tau \geq k\}} E(X_k^2 \mid \mathcal{F}_{k-1}) \right) \\
= E\left( \sum_{k=1}^{\tau \wedge n} E(X_k^2 \mid \mathcal{F}_{k-1}) \right) \leq K.
\]

By Jensen’s inequality \((E|S_n^\tau|)^2 \leq E(S_n^\tau)^2 \leq K^2\). As a consequence \( S^\tau \) is a martingale that is bounded in \( L^1 \). By the martingale convergence theorem, it converges a.s. to an integrable limit \( S_\infty \), say. Thus \( S_n \) converges a.s. on the event \( \tau = \infty \), in other words, on the event \( \sum_{k=1}^{\infty} E(X_k^2 \mid \mathcal{F}_{k-1}) \leq K \}. Let \( K \uparrow \infty \). QED

Without proof we will now recall a simple but effective lemma.

**Lemma 2.2.21 (Kronecker Lemma)** Let \( \{x_n\}_{n \geq 1} \) be a sequence of real numbers such that \( \sum_n x_n \) converges. Let \( \{b_n\}_n \) be a non-decreasing sequence of positive constants with \( b_n \uparrow \infty \) as \( n \to \infty \). Then \( b_n^{-1} \sum_{k=1}^{n} b_k x_k \to 0 \), as \( n \to \infty \).

This allows to formulate the following version of a martingale Law of Large Numbers.

**Theorem 2.2.22** Let \( (S_n = \sum_{k=1}^{n} X_k)_{n \geq 1} \) be a martingale with respect to the filtration \( (\mathcal{F}_n)_{n \geq 1} \). Then \( \sum_{k=1}^{n} X_k/k \) converges a.s. on the set \( \{\sum_{k=1}^{\infty} k^{-2}E(X_k^2 \mid \mathcal{F}_{k-1}) < \infty \} \). Hence \( S_n/n \to 0 \) a.s. on the set \( \{\sum_{k=1}^{\infty} k^{-2}E(X_k^2 \mid \mathcal{F}_{k-1}) < \infty \} \).

**Proof.** Combine Theorem 2.2.6 and Lemma 2.2.21.

### 2.3 Continuous-time martingales

In this section we consider general martingales indexed by a subset \( T \) of \( \mathbb{R}_+ \). If the martingale \( M = (M_t)_{t \geq 0} \) has ‘nice’ sample paths, for instance they are right-continuous, then \( M \) can be ‘approximated’ accurately by a discrete-time martingale. Simply choose a countable dense subset \( \{t_n\} \) of the index set \( T \) and compare the continuous-times martingale \( M \) with the discrete-time martingale \( (M_{t_n})_n \). This simple idea allows to transfer many of the discrete-time results to the continuous-time setting.

#### 2.3.1 Upcrossings in continuous time

For a continuous-time process \( X \) we define the number of upcrossings of the interval \([a, b]\) in the bounded set of time points \( T \subset \mathbb{R}_+ \) as follows. For a finite set \( F = \{t_1, \ldots, t_n\} \subset T \) we define \( U_F[a, b] \) as the number of upcrossings of \([a, b]\) of the discrete-time process \((X_{t_i})_{i=1,\ldots,n}\) (see Definition 2.2.11). We put

\[
U_T[a, b] = \sup\{U_F[a, b] \mid F \subseteq T, F \text{ finite}\}.
\]
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Doob’s upcrossing lemma has the following extension.

**Lemma 2.3.1.** Let $M$ be a supermartingale and let $T \subseteq \mathbb{R}_+$ be a countable, bounded set. Then for all $a < b$, the number of upcrossings $U_T[a, b]$ of the interval $[a, b]$ by $M$ satisfies

$$(b - a)E_U[a, b] \leq \sup_{t \in T} E(M_t - a)^-.$$  

**Proof.** Let $T_n$ be a nested sequence of finite sets, such that $U_T[a, b] = \lim_{n \to \infty} U_{T_n}[a, b]$. For every $n$, the discrete-time upcrossing inequality states that

$$(b - a)E_{U_{T_n}}[a, b] \leq E(M_{t_n} - a)^-,$$

where $t_n$ is the largest element of $T_n$. By the conditional version of Jensen’s inequality the process $(M - a)^-$ is a submartingale (see Example 2.1.3). In particular, the function $t \mapsto E(M_{t_n} - a)^-$ is increasing, so

$$E(M_{t_n} - a)^- = \sup_{t \in T_n} E(M_t - a)^-.$$  

So, for every $n$ we have the inequality

$$(b - a)E_{U_{T_n}}[a, b] \leq \sup_{t \in T_n} E(M_t - a)^-.$$  

The proof is completed by letting $n$ tend to infinity. QED

Hence $U_T[a, b]$ is a.s. finite! By Doob’s upcrossing Lemma 2.2.12 $U_T[a, b]$ is $\mathcal{F}_t$-measurable if $t = \sup\{s \mid s \in T\}$.

### 2.3.2 Regularisation

We always consider the processes under consideration to be defined on an underlying filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in T}, P)$. Remind that $\mathcal{F}_\infty = \sigma(\mathcal{F}_t, t \in T)$. We will assume that $T = \mathbb{R}_+$. For shorthand notation, if we write $\lim_{q \uparrow t}$, we mean the limit along non-increasing (non-decreasing) rational sequences converging to $t$. The same holds for $\lim\sup / \lim\inf$.

**Theorem 2.3.2** Let $M$ be a supermartingale. Then there exists a set $\Omega^*_s \in \mathcal{F}_s$ of probability 1, such that for all $\omega \in \Omega^*_s$ the limits

$$\lim_{q \uparrow t} M_q(\omega) \quad \text{and} \quad \lim_{q \downarrow t} M_q(\omega)$$  

exist and are finite for every $t \in (0, s]$ and $t \in [0, s)$ respectively, for any $s \leq \infty$.

Is the set of discontinuities of each path $M(\omega) = (M_t(\omega))_{t \in \mathbb{R}_+}, \omega \in \Omega^*$, at most countable? This is not (yet) clear.

**Proof.** We give the proof for $s = \infty$. Fix $n \in \mathbb{Z}_+$. Let $a < b$, $a, b \in \mathbb{Q}$. By virtue of Lemma 2.3.1 there exists a set $\Omega_{n,a,b} \in \mathcal{F}_n$, of probability 1, such that

$$U_{[0,n] \cap \mathbb{Q}[a,b]}(\omega) < \infty, \quad \text{for all } \omega \in \Omega_{n,a,b}.$$
Put
\[ \Omega_n = \bigcap_{a<b, a,b \in \mathbb{Q}} \Omega_{n,a,b}. \]
Then \( \Omega_n \in \mathcal{F}_n \). Let now \( t < n \) and suppose that
\[ \lim_{q \downarrow t} M_q(\omega) \]
does not exist for some \( \omega \in \Omega_n \). Then there exists \( a < b, a, b \in \mathbb{Q} \), such that
\[ \liminf_{q \downarrow t} M_q(\omega) < a < b < \limsup_{q \downarrow t} M_q(\omega). \]
Hence \( U_{[0,n] \cap \mathbb{Q}}[a,b](\omega) = \infty \), a contradiction. It follows that \( \lim_{q \downarrow t} M_q(\omega) \) exists for all \( \omega \in \Omega_n \) and all \( t \in [0,n) \).

A similar argument holds for the left limits: \( \lim_{q \uparrow t} M_q(\omega) \) exists for all \( \omega \in \Omega_n \), for all \( t \in (0,n] \). It follows that on \( \Omega' = \cap_n \Omega_n \) these limits exist in \([-\infty, \infty]\) for all \( t > 0 \) in case of left limits and for all \( t \geq 0 \) in case of right limits. Note that \( \Omega' \in \mathcal{F}_\infty \) and \( \mathbb{P}(\Omega') = 1 \).

We still have to show that the limits are in fact finite. Fix \( t \in T \), \( n > t \). Let \( Q_n = [0,n] \cap \mathbb{Q} \) and let \( Q_{m,n} \) be a nested sequence of finitely many rational numbers increasing to \( Q_n \), all containing \( 0 \) and \( n \). Then \( (M_s)_{s \in Q_{m,n}} \) is a discrete-time supermartingale. By virtue of Theorem 2.2.10
\[ \lambda \mathbb{P}(\max_{s \in Q_{m,n}} |M_s| > 3\lambda) \leq 4\mathbb{E}|M_0| + 3\mathbb{E}|M_n|. \]
Letting \( m \to \infty \) and then \( \lambda \to \infty \), by virtue of the monotone convergence theorem for sets
\[ \sup_{s \in Q_n} |M_s| < \infty, \quad \text{a.s.} \]
This implies that the limits are finite.

Put \( \Omega''_n = \{ \omega \mid \sup_{s \in Q_n} |M_s|(|\omega| < \infty) \} \). By the above, \( \Omega''_n \in \mathcal{F}_n \) and \( \mathbb{P}(\Omega''_n) = 1 \). Hence, \( \Omega'' := \cap_n \Omega''_n \) is a set of probability 1, belonging to \( \mathcal{F}_\infty \). Finally, set \( \Omega^* = \Omega'' \cap \Omega' \). This is a set of probability 1, belonging to \( \mathcal{F}_\infty \). QED

**Corollary 2.3.3** There exists an \( \mathcal{F}_\infty \)-measurable set \( \Omega^* \), \( \mathbb{P}(\Omega^*) = 1 \), such that every sample path of a right-continuous supermartingale is cadlag on \( \Omega^* \).

**Proof.** See Exercise 2.22. QED

Our aim is now to construct a modification of a supermartingale that is a supermartingale with a.s. cadlag sample paths itself, under suitable conditions. To this end read LN §1.6, definitions 1.6.2 (right-continuity of a filtration) and 1.6.3 (usual conditions)

Given a supermartingale \( M \), define for every \( t \geq 0 \)
\[ M^+_{t+}(\omega) = \begin{cases} \lim_{q \downarrow t, q \in \mathbb{Q}} M_q(\omega), & \text{if this limit exists and is finite} \\ 0, & \text{otherwise} \end{cases} \]
The random variables \( M^+_{t+} \) are well-defined by Theorem 2.3.2. By inspection of the proof of this theorem, one can check that \( M^+_{t+} \) is \( \mathcal{F}_{t+} \)-measurable.

We have the following result concerning the process \( (M^+_{t+})_{t \geq 0} \).
### Lemma 2.3.4
Let $M$ be a supermartingale.

i) Then $\mathbb{E}|M_t^+| < \infty$ for every $t$ and

$$\mathbb{E}(M_t^+ | \mathcal{F}_t) \leq M_t, \quad \text{a.s.}$$

If in addition $t \to \mathbb{E}M_t$ is right-continuous, then this inequality is an equality.

ii) The process $(M_t^+)_{t \geq 0}$ is a supermartingale with respect to the filtration $(\mathcal{F}_t^+)_{t \geq 0}$ and it is a martingale if $M$ is a martingale.

**Proof.** Fix $t \geq 0$. Let $q_n \downarrow t$ be a sequence of rational numbers decreasing to $t$. Then the process $(M_{q_n})_n$ is a backward discrete-time supermartingale like we considered in Theorem 2.2.16, with $\sup_n \mathbb{E}M_{q_n} \leq \mathbb{E}M_t < \infty$. By that theorem, $M_{q_n}$ is integrable, and $M_{q_n} \xrightarrow{L^1} M_{t^+}$. As in the proof of Theorem 2.2.14, $L^1$-convergence allows to take the limit $n \to \infty$ in the inequality

$$\mathbb{E}(M_{q_n} | \mathcal{F}_t) \leq M_t \quad \text{a.s.}$$

yielding

$$\mathbb{E}(M_{t^+} | \mathcal{F}_t) \leq M_t \quad \text{a.s.}$$

$L^1$ convergence also implies that $\mathbb{E}M_{q_n} \to \mathbb{E}M_{t^+}$. So, if $s \to \mathbb{E}M_s$ is right-continuous, then $\mathbb{E}M_{t^+} = \lim_{n \to \infty} \mathbb{E}M_{q_n} = \mathbb{E}M_t$. But this implies (cf. Exercise 2.7 b) that $\mathbb{E}(M_{t^+} | \mathcal{F}_t) = M_t$ a.s.

To prove the final statement, let $s < t$ and let $q_n^s \leq t$ be a sequence of rational numbers decreasing to $s$. Then

$$\mathbb{E}(M_{t^+} | \mathcal{F}_{q_n^s}) = \mathbb{E}(\mathbb{E}(M_{t^+} | \mathcal{F}_t) | \mathcal{F}_{q_n^s}) \leq \mathbb{E}(M_t | \mathcal{F}_{q_n^s}) \leq M_{q_n^s} \quad \text{a.s.}$$

with equality if $M$ is a martingale. The right-hand side of inequality converges to $M_{s^+}$ as $n \to \infty$. The process $\mathbb{E}(M_{t^+} | \mathcal{F}_{q_n^s})$ is a backward martingale satisfying the conditions of Theorem 2.2.16. Hence, $\mathbb{E}(M_{t^+} | \mathcal{F}_{q_n^s})$ converges to $\mathbb{E}(M_{t^+} | \mathcal{F}_{s^+})$ a.s. QED

We can now prove the main regularisation theorem for supermartingales with respect to filtrations satisfying the usual conditions.

The idea is the following. In essence we want ensure that all sample paths are c.adl.a priori (that the cadlag paths are a measurable set w.r.t to each $\sigma$-algebra $\mathcal{F}_t$). In view of Corollary 2.3.3 this requires to complete $\mathcal{F}_0$ with all $\mathcal{P}$-null sets in $\mathcal{F}_\infty$. On the other hand, we want to make out of $(M_{t^+})_t$ a cadlag modification of $M$. This is guaranteed if the filtration involved is right-continuous.

Can one enlarge a given filtration to obtain a filtration satisfying the usual conditions? If yes, can this procedure destroy properties of interest - like the supermartingale property, independence properties? For some details on this issue see BN §9.

### Theorem 2.3.5 (Doob’s regularity theorem)
Let $M$ be a supermartingale with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$ satisfying the usual conditions. Then $M$ has a cadlag modification $\tilde{M}$ (such that $\{M_t - \tilde{M}_t \neq 0\}$ is a null set contained in $\mathcal{F}_0$) if and only if $t \to \mathbb{E}M_t$ is right-continuous. In that case $\tilde{M}$ is a supermartingale with respect to $(\mathcal{F}_t)_{t \geq 0}$ as well. If $M$ is a martingale then $\tilde{M}$ is a martingale.
Proof. ‘$\Leftarrow$’ By Theorem 2.3.2 and the fact that $(\mathcal{F}_t)$ satisfies the usual conditions, there exists an event $\Omega^* \in \mathcal{F}_0$ (!) of probability 1, on which the limits

$$M_{t^-} = \lim_{q \downarrow t} M_q, \quad M_{t^+} = \lim_{q \uparrow t} M_q$$

exist for every $t$. Define the process $\tilde{M}$ by $\tilde{M}_t(\omega) = M_{t^+}(\omega) 1_{\{\Omega^*\}}(\omega)$ for $t \geq 0$. Then $\tilde{M}_t = M_{t^+}$ a.s. and they differ at most on a null-set contained in $\mathcal{F}_0$. Since $M_{t^+}$ is $\mathcal{F}_{t^+}$-measurable, we have that $\tilde{M}_t$ is $\mathcal{F}_{t^+}$-measurable. It follows that $\tilde{M}_t = \mathbb{E}(M_{t^+} | \mathcal{F}_{t^+})$ a.s. By right-continuity of the filtration, right-continuity of the map $t \to \mathbb{E}M_t$ and the preceding lemma, we get that a.s. $\tilde{M}_t = \mathbb{E}(M_{t^+} | \mathcal{F}_{t^+}) = \mathbb{E}(M_{t^+} | \mathcal{F}_t) = M_t$ a.s. In other words, $\tilde{M}$ is a modification of $M$. Right-continuity of the filtration implies further that $\tilde{M}$ is adapted to $(\mathcal{F}_t)$. The process $\tilde{M}$ is cadlag as well as a supermartingale (see Exercise 2.23).

‘$\Rightarrow$’ see Exercise 2.24. QED

**Corollary 2.3.6** A martingale with respect to a filtration that satisfies the usual conditions has a cadlag modification, which is a martingale w.r.t the same filtration.

We will next give two example showing what can go wrong without right-continuity of the filtration, or without continuity of the expectation as a function of time.

**Example**

Let $\Omega = \{-1, 1\}$, $\mathcal{F}_t = \{\Omega, \emptyset\}$ for $t \leq 1$ and $\mathcal{F}_t = \{\Omega, \emptyset, \{1\}, \{-1\}\}$ for $t > 1$. Let $\mathbb{P}(\{1\}) = \mathbb{P}(\{-1\}) = 1/2$.

Note that $\mathcal{F}_t$ is *not* right-continuous, since $\mathcal{F}_1 \neq \mathcal{F}_{1+}$! Define

$$Y_t(\omega) = \begin{cases} 0, & t \leq 1 \\ \omega, & t > 1 \end{cases}, \quad X_t(\omega) = \begin{cases} 0, & t < 1 \\ \omega, & t \geq 1 \end{cases}.$$ 

Now, $Y = (Y_t)_t$ is a martingale, but it is not right-continuous, whereas $X = (X_t)_t$ is a right-continuous process. One does have that $\mathbb{E}Y_t = 0$ is a right-continuous function of $t$.

Moreover, $Y_{t^+} = X_t$ and $\mathbb{P}(X_1 = Y_1) = 0$. Hence $X$ is not a cadlag modification of $Y$, and in particular $Y$ cannot have a cadlag modification.

By Lemma 2.3.4 it follows that $\mathbb{E}(X_t | \mathcal{F}_t) = Y_t$, so that $X$ cannot be a martingale w.r.t the filtration $(\mathcal{F}_t)_t$. By the same lemma, $X$ is a right-continuous martingale, w.r.t to $(\mathcal{F}_{t^+})_t$. On the other hand, $Y$ is *not* a martingale w.r.t. to $(\mathcal{F}_{t^+})_t$!

**Example**

Let $\Omega = \{-1, 1\}$, $\mathcal{F}_t = \{\Omega, \emptyset\}$ for $t \leq 1$ and $\mathcal{F}_t = \{\Omega, \emptyset, \{1\}, \{-1\}\}$ for $t \geq 1$. Let $\mathbb{P}(\{1\}) = \mathbb{P}(\{-1\}) = 1/2$. Define

$$Y_t(\omega) = \begin{cases} 0, & t \leq 1 \\ 1 - t, & \omega = 1, t > 1 \\ -1, & \omega = -1, t > 1 \end{cases}, \quad X_t(\omega) = \begin{cases} 0, & t < 1 \\ 1 - t, & \omega = 1, t \geq 1 \\ -1, & \omega = -1, t \geq 1 \end{cases}.$$ 

In this case the filtration $(\mathcal{F}_t)_t$ is right-continuous and $Y$ and $X$ are both supermartingales w.r.t $(\mathcal{F}_t)_t$. Furthermore $X_t = \lim_{q \uparrow t} Y_q$ for $t \geq 0$, but $\mathbb{P}(X_1 = Y_1) = 0$ and hence $X$ is not a modification of $Y$. 

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2.3.3 Convergence theorems

In view of the results of the previous section, we will only consider everywhere right-continuous martingales from this point on. Under this assumption, many of the discrete-time theorems can be generalised to continuous time.

**Theorem 2.3.7** Let \( M \) be a right-continuous supermartingale that is bounded in \( L^1 \). Then \( M_t \) converges a.s. to a finite \( F_\infty \)-measurable limit \( M_\infty \), as \( t \to \infty \), with \( \mathbb{E}|M_\infty| < \infty \).

*Proof.* The first step to show is that we can restrict to take a limit along rational time-sequences. In other words, that \( M_t \overset{a.s.}{\to} M_1 \) as \( t \to \infty \) if and only if

\[
\lim_{q \to \infty} M_q = M_1 \quad \text{a.s.} \tag{2.3.1}
\]

To prove the non-trivial implication in this assertion, assume that (2.3.1) holds. Fix \( \varepsilon > 0 \) and \( \omega \in \Omega \) for which \( M_q(\omega) \to M_\infty(\omega) \). Then there exists a number \( a = a_{\omega, \varepsilon} > 0 \) such that \( |M_q(\omega) - M_\infty(\omega)| < \varepsilon \) for all \( q > a \). Now let \( t > a \) be arbitrary. Since \( M \) is right-continuous, there exists \( q^* > t \) such that \( |M_{q^*}(\omega) - M_t(\omega)| < \varepsilon \). By the triangle inequality, it follows that

\[
|M_t(\omega) - M_\infty(\omega)| \leq |M_{q^*}(\omega) - M_\infty(\omega)| + |M_t(\omega) - M_{q^*}(\omega)| < 2\varepsilon.
\]

This proves that \( M_t(\omega) \to M_\infty(\omega), \ t \to \infty \).

To prove convergence to a finite \( F_\infty \)-measurable, integrable limit, we may assume that \( M \) is indexed by the countable set \( Q_+ \). The proof can now be finished by arguing as in the proof of Theorem 2.2.13, replacing Doob’s discrete-time upcrossing inequality by Lemma 2.3.1.

**QED**

**Corollary 2.3.8** A non-negative, right-continuous supermartingale \( M \) converges a.s. as \( t \to \infty \), to a finite, integrable, \( F_\infty \)-measurable random variable.

*Proof.* Simple consequence of Theorem 2.3.7. **QED**

The following continuous-time extension of Theorem 2.2.14 can be derived by reasoning as in discrete-time. The only slight difference is that for a continuous-time process \( X \) \( L^1 \)-convergence of \( X_t \) as \( t \to \infty \) need not imply that \( X \) is UI.

**Theorem 2.3.9** Let \( M \) be a right-continuous supermartingale that is bounded in \( L^1 \).

i) If \( M \) is uniformly integrable, then \( M_t \to M_\infty \) a.s. and in \( L^1 \), and

\[
\mathbb{E}(M_\infty | \mathcal{F}_t) \leq M_t \quad \text{a.s.}
\]

with equality if \( M \) is a martingale.

ii) If \( M \) is a martingale and \( M_t \to M_\infty \) in \( L^1 \) as \( t \to \infty \), then \( M \) is uniformly integrable.

*Proof.* See Exercise 2.25. **QED**
2.3.4 Inequalities

Dobb’s submartingale inequality and \(L^p\)-inequality are very easily extended to the setting of general right-continuous martingales.

**Theorem 2.3.10 (Doob’s submartingale inequality)** Let \(M\) be a right-continuous submartingale. Then for all \(\lambda > 0\) and \(t \geq 0\)

\[
\mathbb{P}\{\sup_{s \leq t} M_s \geq \lambda\} \leq \frac{1}{\lambda} \mathbb{E}[M_t].
\]

**Proof.** Let \(T\) be a countable, dense subset of \([0, t]\) and choose an increasing sequence of finite subsets \(T_n \subseteq T\), \(0, t \in T_n\) for every \(n\) and \(T_n \uparrow T\) as \(n \to \infty\). By right-continuity of \(M\) we have that

\[
\sup_{n} \max_{s \in T_n} M_s = \sup_{s \in T} M_s = \sup_{s \in [0, t]} M_s.
\]

This implies that \(\{\max_{s \in T_n} M_s > c\} \uparrow \{\sup_{s \in T} M_s > c\}\) and so by monotone convergence of sets \(\mathbb{P}\{\max_{s \in T_n} M_s > c\} \uparrow \mathbb{P}\{\sup_{s \in T} M_s > c\}\). By the discrete-time version of the submartingale inequality for each \(m > 0\) sufficiently large

\[
\mathbb{P}\{\sup_{s \in [0, t]} M_s > \lambda - \frac{1}{m}\} = \mathbb{P}\{\sup_{s \in T} M_s > \lambda - \frac{1}{m}\} = \lim_{n \to \infty} \mathbb{P}\{\max_{s \in T_n} M_s > \lambda - \frac{1}{m}\} \leq \frac{1}{\lambda - 1/m} \mathbb{E}[M_t].
\]

Let \(m\) tend to infinity. QED

By exactly the same reasoning, we can generalise the \(L^p\)-inequality to continuous time.

**Theorem 2.3.11 (Doob’s \(L^p\)-inequality)** Let \(M\) be a right-continuous martingale or a right-continuous, nonnegative submartingale. Then for all \(p > 1\) and \(t \geq 0\)

\[
\mathbb{E}\left(\sup_{s \leq t} |M_s|^p\right) \leq \left(\frac{p}{p - 1}\right)^p \mathbb{E}[|M_t|^p].
\]

2.3.5 Optional sampling

We will now discuss the continuous-time version of the optional stopping theorem.

**Theorem 2.3.12 (Optional sampling theorem)** Let \(M\) be a right-continuous, uniformly integrable supermartingale. Then for all stopping times \(\sigma \leq \tau\) we have that \(M_\tau\) and \(M_\sigma\) are integrable and

\[
\mathbb{E}(M_\tau | \mathcal{F}_\sigma) \leq M_\sigma \quad \text{a.s.}
\]

with equality if \(M\) is martingale.
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Proof. By Lemma 1.6.17 there exist stopping times \( \sigma_n \) and \( \tau_n \) taking only finitely many values, such that \( \sigma_n \leq \tau_n \), and \( \sigma_n \downarrow \sigma \), \( \tau_n \downarrow \tau \).

By the discrete-time optional sampling theorem applied to the supermartingale \((M_{k/2^n})_{k \in \mathbb{Z}_+}\) (it is uniformly integrable!)

\[
E(M_{\tau_n} | \mathcal{F}_{\sigma_n}) \leq M_{\sigma_n} \quad \text{a.s.}
\]

Since \( \sigma \leq \sigma_n \) it holds that \( \mathcal{F}_{\sigma} \subseteq \mathcal{F}_{\sigma_n} \). It follows that

\[
E(M_{\tau_n} | \mathcal{F}_{\sigma}) = E\left(E(M_{\tau_n} | \mathcal{F}_{\sigma_n}) | \mathcal{F}_{\sigma}\right) \leq E(M_{\sigma_n} | \mathcal{F}_{\sigma}) \quad \text{a.s.} \tag{2.3.2}
\]

Similarly

\[
E(M_{\tau_n} | \mathcal{F}_{\tau_n+1}) \leq M_{\tau_n+1} \quad \text{a.s.}
\]

Hence, \((M_{\tau_n})_n\) is a ‘backward’ supermartingale in the sense of the Lévy-Doob downward theorem 2.2.16. Since \( \sup_n EM_{\tau_n} \leq EM_0 \), this theorem implies that \((M_{\tau_n})_n\) is uniformly integrable and it converges a.s. and in \(L^1\). By right-continuity \(M_{\tau_n} \to M_\tau\) a.s. Hence \(M_{\tau_n} \to M_\tau\) in \(L^1\). Similarly \(M_{\sigma_n} \to M_\sigma\) in \(L^1\). Now take \(A \in \mathcal{F}_{\sigma}\). By equation (2.3.2) it holds that

\[
\int_A M_{\tau_n} dP \leq \int_A M_{\sigma_n} dP.
\]

By \(L^1\)-convergence, this yields

\[
\int_A M_\tau dP \leq \int_A M_\sigma dP,
\]

if we let \(n\) tend to infinity. This completes the proof. QED

So far we have not yet addressed the question whether stopped (sub-, super-) martingales are (sub-, super-) martingales. The next theorem prepares the way to prove this.

Theorem 2.3.13 A right-continuous, adapted process \(M\) is a supermartingale (resp. a martingale) if and only if for all bounded stopping times \(\tau, \sigma\) with \(\sigma \leq \tau\), the random variables \(M_\tau\) and \(M_\sigma\) are integrable and \(EM_\tau \leq EM_\sigma\) (resp. \(EM_\tau = EM_\sigma\)).

Proof. Suppose that \(M\) is a supermartingale. Since \(\tau\) is bounded, there exists a constant \(K > 0\) such that \(\tau \leq K\) a.s.

As in the construction in the proof of Lemma 1.6.17 there exist stopping times \(\tau_n \downarrow \tau\) and \(\sigma_n \downarrow \sigma\) that are bounded by \(K\) and take finitely many values. In particular \(\tau_n, \sigma_n \in D_n = \{K \cdot k \cdot 2^{-n}, k = 0, 1, \ldots, 2^n\}\). Note that \((D_n)_n\) is an increasing sequence of sets.

By bounded optional stopping for discrete-time martingales, we have that \(E(M_{\tau_n} | \mathcal{F}_{\tau_n+1}) \leq M_{\tau_n+1}\) a.s. (consider \(M\) restricted to the discrete time points \(\{k \cdot 2^{-(n+1)}, k \in \mathbb{Z}_+\}\)). We can apply the Lévy-Doob downward theorem 2.2.16, to obtain that \((M_{\tau_n})_n\) is a uniformly integrable supermartingale, converging a.s. and in \(L^1\) to an integrable limit as in the proof of the previous theorem. By right-continuity the limit is \(M_\tau\). Analogously, we obtain that \(M_\sigma\) is integrable.

Bounded optional stopping for discrete-time supermartingales similarly yields that \(E(M_{\tau_n} | M_{\sigma_n}) \leq M_{\sigma_n}\) a.s. Taking expectations and using \(L^1\)-convergence proves that \(EM_\tau \leq EM_\sigma\).

The reverse statement is proved by arguing as in the proof of Theorem 2.2.5. Let \(s \leq t\) and let \(A \in \mathcal{F}_s\). Choose stopping times \(\sigma = s\) and \(\tau = 1_{\{A\}^c} + 1_{\{A^c\}} s\). QED
Corollary 2.3.14 If \( M \) is a right-continuous (super)martingale and \( \tau \) is an \((F_t)\) stopping time, then the stopped process is a (super)martingale as well.

Proof. Note that \( M^\tau \) is right-continuous. By Lemmas 1.6.14 and 1.6.16 it is adapted.

Let \( \sigma \leq \xi \) be bounded stopping times. By applying the previous theorem to the super-martingale \( M \), and using that \( \sigma \wedge \tau, \xi \wedge \tau \) are bounded stopping times, we find that

\[
E M^\tau_{\sigma} = E M_{\sigma} \leq E M_{\tau \wedge \xi} = E M^\tau_{\xi}.
\]

Since \( \sigma \) and \( \xi \) were arbitrary bounded stopping times, another application of the previous theorem yields the desired result. QED

Just as in discrete time the assumption of uniform integrability is crucial for the optional sampling theorem. If this condition is dropped, we only have an inequality in general.

Theorem 2.2.19 carries over to continuous time by using the same arguments as in the proof of Theorem 2.3.12.

Theorem 2.3.15 Let \( M \) be a right-continuous non-negative supermartingale and let \( \sigma \leq \tau \) be stopping times. Then

\[
E(M_{\tau} \mid F_{\sigma}) \leq M_{\sigma}, \quad \text{a.s.}
\]

A consequence of this result is that non-negative right-continuous supermartingales stay at zero once they have hit it.

Corollary 2.3.16 Let \( M \) be a non-negative, right-continuous supermartingale and define \( \tau = \inf\{t \mid M_t = 0 \text{ or } M_{t-} = 0\} \). Then \( P(M_{\tau+q1_{\tau<\infty}} = 0, t \geq 0) = 1 \), where \( M_{t-} = \lim sup_{s \uparrow t} M_s \).

Proof. Positive supermartingales are bounded in \( L^1 \). By Theorem 2.3.7, \( M \) converges a.s. to an integrable limit \( M_{\infty} \) say.

Note that \( \lim sup_{s \uparrow t} M_s = 0 \) implies that \( \lim sup_{s \uparrow t} M_s \) exists and equals 0. Hence by right-continuity \( \tau(\omega) \leq t \) if and only if \( \inf\{M_q(\omega) \mid q \in Q \cap [0, t]\} = 0 \) or \( M_t(\omega) = 0 \). Hence \( \tau \) is a stopping time. Define

\[
\tau_n = \inf\{t \mid M_t < n^{-1}\}.
\]

Then \( \tau_n \) is a stopping time with \( \tau_n \leq \tau \). Furthermore, for all \( q \in Q_+ \), we have that \( \tau + q \) is a stopping time. Then by the foregoing theorem

\[
E(M_{\tau+q} \leq E M_{\tau_n} \leq \frac{1}{n} P(\tau_n < \infty) + E M_{\infty} 1_{\{\tau_n = \infty}\}.
\]

On the other hand, since \( \tau_n = \infty \) implies \( \tau + q = \infty \) we have

\[
E M_{\infty}(1_{\{\tau_n = \infty\}}) \leq E M_{\infty}(1_{\{\tau + q = \infty\}}).
\]

Combination yields for all \( n \in \mathbb{Z}_+ \) and all \( q \in Q_+ \)

\[
E M_{\tau+q} 1_{\{\tau + q < \infty\}} = E M_{\tau+q} - E M_{\infty} 1_{\{\tau + q = \infty\}} \leq \frac{1}{n} P(\tau_n < \infty).
\]
Taking the limit \( n \to \infty \) yields \( \mathbb{EM}_{r+q}1_{\{\tau + q < \infty\}} = 0 \) for all \( q \in \mathbb{Q}_+ \). By non-negativity of \( M \) we get that \( M_{r+q}1_{\{\tau + q < \infty\}} = 0 \) a.s. for all \( q \in \mathbb{Q}_+ \). But then also \( \mathbb{P}\{\cap_q\{M_{r+q} = 0\} \cup \{\tau < \infty\}\} = 1 \). By right continuity

\[
\cap_q\{M_{r+q} = 0\} \cap \{\tau < \infty\} = \{M_{r+t} = 0, \tau < \infty, t \geq 0\}.
\]

Note that the set where \( M_{r+t} = 0 \) belongs to \( \mathcal{F}_\infty \)!

QED

### 2.4 Applications to Brownian motion

In this section we apply the developed theory to the study of Brownian motion.

#### 2.4.1 Quadratic variation

The following result extends the result of Exercise 1.15 of Chapter 1.

**Theorem 2.4.1** Let \( W \) be a Brownian motion and fix \( t > 0 \). For \( n \in \mathbb{Z}_+ \), let \( \pi_n \) be a partition of \([0, t]\) given by \( 0 = t^n_0 \leq t^n_1 \leq \cdots \leq t^n_{k_n} = t \) and suppose that the mesh \( |\pi_n| = \max_k |t^n_k - t^n_{k-1}| \) tends to zero as \( n \to \infty \). Then

\[
\sum_k (W^n_k - W^n_{k-1})^2 \xrightarrow{L^2} t, \quad t \to \infty.
\]

If the partitions are nested we have

\[
\sum_k (W^n_k - W^n_{k-1})^2 \xrightarrow{a.s.} t, \quad t \to \infty.
\]

**Proof.** For the first statement see Exercise 1.15 in Chapter 1. To prove the second one, denote the sum by \( X_n \) and put \( \mathcal{F}_n = \sigma(X_n, X_{n+1}, \ldots) \). Then \( \mathcal{F}_{n+1} \subset \mathcal{F}_n \) for every \( n \in \mathbb{Z}_+ \). Now suppose that we can show that \( \mathbb{E}(X_n | \mathcal{F}_{n+1}) = X_{n+1} \) a.s. Then, since \( \sup\mathbb{E}X_n < \infty \), the Lévy-Doob downward theorem 2.2.16 implies that \( X_n \) converges a.s. to a finite limit \( X_\infty \). By the first statement of the theorem the \( X_n \) converge in probability to \( t \). Hence, we must have \( X_\infty = t \) a.s.

So it remains to prove that \( \mathbb{E}(X_n | \mathcal{F}_{n+1}) = X_{n+1} \) a.s. Without loss of generality, we assume that the number of elements of the partition \( \pi_n \) equals \( n \). In that case, there exists a sequence \( t_n \) such that the partition \( \pi_n \) has the numbers \( t_1, \ldots, t_n \) as its division points: the point \( t_n \) is added to \( \pi_{n-1} \) to form the next partition \( \pi_n \). Now fix \( n \) and consider the process \( W' \) defined by

\[
W'_s = W_{s \wedge t_{n+1}} - (W_s - W_{s \wedge t_{n+1}}).
\]

By Exercise 1.12 of Chapter 1, \( W' \) is again a BM. For \( W' \), denote the analogous sums \( X_k \) by \( X'_k \). Then it is easily seen for \( k \geq n+1 \) that \( X'_k = X_k \). Moreover, it holds that \( X'_n - X'_{n+1} = X_{n+1} - X_n \) (check!). Since both \( W \) and \( W' \) are BM’s, the sequences \( (X_1, X_2, \ldots) \) and \( (X'_1, X'_2, \ldots) \) have the same distribution. It follows that a.s.

\[
\begin{align*}
\mathbb{E}(X_n - X_{n+1} | \mathcal{F}_{n+1}) &= \mathbb{E}(X'_n - X'_{n+1} | X'_{n+1}, X'_{n+2}, \ldots) \\
&= \mathbb{E}(X'_n - X'_{n+1} | X_{n+1}, X_{n+2}, \ldots) \\
&= \mathbb{E}(X_{n+1} - X_n | X_{n+1}, X_{n+2}, \ldots) \\
&= -\mathbb{E}(X_{n+1} - X_n | \mathcal{F}_{n+1}).
\end{align*}
\]
This implies that $E(X_n - X_{n+1} \mid \mathcal{F}_{n+1}) = 0$ a.s. \[\text{QED}\]

A real-valued function $f$ is said to be of finite variation on an interval $[a, b]$, if there exists a finite number $K > 0$, such that for every finite partition $a = t_0 < \cdots < t_n = b$ of $[a, b]$ it holds that

$$\sum_k |f(t_k) - f(t_{k-1})| < K.$$ 

Roughly speaking, this means that the graph of the function $f$ on $[a, b]$ has finite length.

Theorem 2.4.1 shows that the sample paths of BM have positive, finite quadratic variation. This has the following consequence.

**Corollary 2.4.2** *Almost every sample path of BM has unbounded variation on every interval.*

**Proof.** Fix $t > 0$. Let $\pi_n$ be nested partitions of $[0, t]$ given by $0 = t_0^n \leq t_1^n \leq \cdots \leq t_k^n = t$. Suppose that the mesh $||\pi_n|| = \max_k |t_k^n - t_{k-1}^n| \to 0$ as $n \to \infty$. Then

$$\sum_k (W_{t_k^n} - W_{t_{k-1}^n})^2 \leq \max_k |W_{t_k^n} - W_{t_{k-1}^n}| \cdot \sum_k |W_{t_k^n} - W_{t_{k-1}^n}|.$$ 

By uniform continuity of Brownian sample paths, the first factor on the right-hand side converges to zero a.s., as $n \to \infty$. Hence, if the Brownian motion would have finite variation on $[0, t]$ with positive probability, then $\sum_k (W_{t_k^n} - W_{t_{k-1}^n})^2$ would converge to 0 with positive probability. This contradicts Theorem 2.4.1. \[\text{QED}\]

### 2.4.2 Exponential Inequality

Let $W$ be a Brownian motion. We have the following exponential inequality for the tail properties of the running maximum of the Brownian motion.

**Theorem 2.4.3** *For every $t \geq 0$ and $\lambda > 0$*

$$P\left\{\sup_{s \leq t} W_s \geq \lambda\right\} \leq e^{-\lambda^2/2t}$$

and

$$P\left\{\sup_{s \leq t} |W_s| \geq \lambda\right\} \leq 2e^{-\lambda^2/2t}$$

**Proof.** For $\alpha > 0$ consider the exponential martingale $M$ defined by $M_t = \exp\{a W_t - a^2 t/2\}$ (see Example 2.1.4). Observe that

$$P\{\sup_{s \leq t} W_s \geq \lambda\} \leq P\{\sup_{s \leq t} M_s \geq e^{a\lambda - a^2 t/2}\}.$$ 

By the submartingale inequality, the probability on the right-hand side is bounded by

$$e^{a^2 t/2 - a\lambda} E M_t = e^{a^2 t/2 - a\lambda} E M_0 = e^{a^2 t/2 - a\lambda}.$$
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The proof of the first inequality is completed by minimising the latter expression in \( a > 0 \). To prove the second one, note that

\[
\mathbb{P}\{\sup_{s \leq t} |W_s| \geq \lambda\} \leq \mathbb{P}\{\sup_{s \leq t} W_s \geq \lambda\} + \mathbb{P}\{\inf_{s \leq t} W_s \leq -\lambda\} = \mathbb{P}\{\sup_{s \leq t} W_s \geq \lambda\} + \mathbb{P}\{\sup_{s \leq t} -W_s \geq \lambda\}.
\]

The proof is completed by applying the first inequality to the BM’s \( W \) and \(-W\). QED

The exponential inequality also follows from the fact that \( \sup_{s \leq t} W_s \overset{d}{=} |W_t| \) for every fixed \( t \). We will prove this equality in distribution in the next chapter.

2.4.3 The law of the iterated logarithm

The law of the iterated logarithm describes how BM oscillates near zero and infinity. In the proof we will need the following simple lemma.

**Lemma 2.4.4** For every \( a > 0 \)

\[
\int_a^\infty e^{-x^2/2} dx \geq \frac{a}{1+a^2}e^{-a^2/2}.
\]

**Proof.** The proof starts from the inequality

\[
\int_a^\infty \frac{1}{x^2}e^{-x^2/2} dx \leq \frac{1}{a^2} \int_a^\infty e^{-x^2/2} dx.
\]

Integration by parts shows that the left-hand side equals

\[
- \int_a^\infty e^{-x^2/2}d\left(\frac{1}{x}\right) = \frac{1}{a}e^{-a^2/2} + \int_a^\infty \frac{1}{x}d(e^{-x^2/2}) = \frac{1}{a}e^{-a^2/2} - \int_a^\infty e^{-x^2/2} dx.
\]

Hence we find that

\[
\left(1 + \frac{1}{a^2}\right) \int_a^\infty e^{-x^2/2} dx \geq \frac{1}{a}e^{-a^2/2}.
\]

This finishes the proof. QED

**Theorem 2.4.5 (Law of the iterated logarithm)** It almost surely holds that

\[
\lim_{t \to 0} \sup_{t \leq 0} \frac{W_t}{\sqrt{2t \log \log 1/t}} = 1, \quad \lim_{t \to 0} \inf_{t \leq 0} \frac{W_t}{\sqrt{2t \log \log 1/t}} = -1,
\]

\[
\lim_{t \to \infty} \sup_{t \geq 0} \frac{W_t}{\sqrt{2t \log \log t}} = 1, \quad \lim_{t \to \infty} \inf_{t \geq 0} \frac{W_t}{\sqrt{2t \log \log t}} = -1.
\]
Proof. It suffices to prove the first statement. The second follows by applying the first to the BM $-W$. The third and fourth statements follow by applying the first two to the BM $t W_{1/t}$ (cf. Theorem 1.4.4).

Put $h(t) = \sqrt{2t \log \log 1/t}$. We will first prove that $\lim sup_{s \leq 0} W(s)/h(s) \leq 1$. Choose two numbers $\theta, \delta \in (0, 1)$. We put

$$
\alpha_n = (1 + \delta)\theta^{-n} h(\theta^n), \quad \beta_n = h(\theta^n)/2.
$$

Use the submartingale inequality applied to the exponential martingale $M_s = \exp\{\alpha_n W_s - \alpha_n^2 s/2\}$:

$$
P \{ \sup_{s \leq 1} (W_s - \alpha_n s/2) \geq \beta_n \} = P \{ \sup_{s \leq 1} M_s \geq e^{\alpha_n \beta_n} \} \leq e^{-\alpha_n \beta_n} EM_1 = e^{-\alpha_n \beta_n} \leq K \theta n^{-(1+\delta)},
$$

for some constant $K_\theta > 0$ that depends on $\theta$ but not on $n$. Applying the Borel-Cantelli lemma yields the existence of a set $\Omega_{\theta, \delta}^{*} \in \mathcal{F}_{\infty}$, $P(\Omega_{\theta, \delta}^{*}) = 1$, such that for each $\omega \in \Omega_{\theta, \delta}^{*}$ there exists $n_\omega$ such that

$$
\sup_{s \leq 1} (W_s(\omega) - \alpha_n s/2) \leq \beta_n,
$$

for $n \geq n_\omega$.

One can verify that $h$ is increasing for $t \in (0, e^{-c}]$ with $c$ satisfying $c = e^{1/c}$. Hence, for $n \geq n_\omega$ and $s \in [\theta^n, \theta^{n-1}]$

$$
W_s(\omega) \leq \frac{\alpha_n s}{2} + \beta_n \leq \frac{\alpha_n \theta^{n-1}}{2} + \beta_n = \left(\frac{1 + \delta}{2\theta} + \frac{1}{2}\right) h(\theta^n) \leq \left(\frac{1 + \delta}{2\theta} + \frac{1}{2}\right) h(s).
$$

It follows that for all $n \geq n_\omega \lor -c/\log \theta$

$$
\sup_{\theta^n s \leq \theta^{n-1}} \frac{W_s(\omega)}{h(s)} \leq \left(\frac{1 + \delta}{2\theta} + \frac{1}{2}\right).
$$

It follows that $\lim sup_{s \leq 0} W_s(\omega)/h(s) \leq \left(\frac{1 + \delta}{2\theta} + \frac{1}{2}\right)$, for $\omega \in \Omega_{\theta, \delta}^{*}$. Write $\Omega_m^{*} = \Omega_{\theta=1-1/m, \delta=1/m}$ and put $\Omega^* = \cap_m \Omega_m^{*}$. Then $\lim sup_{s \leq 0} W_s(\omega)/h(s) \leq 1$ for $\omega \in \Omega^*$.

To prove the reverse inequality, choose $\theta \in (0, 1)$ and consider the events

$$
A_n = \{W_{\theta^n} - W_{\theta^{n+1}} \geq (1 - \sqrt{\theta}) h(\theta^n)\}.
$$

By the independence of the incements of BM, the events $A_n$ are independent. Note that

$$
W_{\theta^n} - W_{\theta^{n+1}} \overset{d}{=} N(0, 1).
$$

Hence,

$$
P \{ A_n \} = P \left\{ \frac{W_{\theta^n} - W_{\theta^{n+1}}}{\sqrt{\theta^n - \theta^{n+1}}} \geq \frac{(1 - \sqrt{\theta}) h(\theta^n)}{\sqrt{\theta^n - \theta^{n+1}}} \right\} = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-x^2/2} dx,
$$

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with

\[ a = \frac{(1 - \sqrt{\theta})h(\theta^n)}{\sqrt{\theta^n} - \theta^{n+1}} = (1 - \sqrt{\theta})\sqrt{\frac{2\log \log \theta^{-n}}{1 - \theta}}. \]

By Lemma 2.4.4 it follows that

\[ \sqrt{2\pi}P\{A_n\} \geq \frac{a}{1 + a^2}e^{-a^2/2}. \]

It is easily seen that the right-hand side is of order

\[ n^{-\frac{(1-\sqrt{\theta})}{1-\theta}} = n^{-\alpha}, \]

with \( \alpha < 1 \). It follows that \( \sum P\{A_n\} = \infty \) and so by the 2d Borel-Cantelli Lemma there exists a set \( \Omega_\theta \in \mathcal{F}_\infty \), \( P\{\Omega_\theta\} = 1 \), such that

\[ W_{\theta^n}(\omega) \geq (1 - \sqrt{\theta})h(\theta^n) + W_{\theta^{n+1}}(\omega) \]

for infinitely many \( n \), for all \( \omega \in \Omega_\theta \). Since \(-W\) is also a BM, the first part of the proof implies the existence of a set \( \Omega_0 \in \mathcal{F}_\infty \), \( P\{\Omega_0\} = 1 \), such that for each \( \omega \in \Omega_0 \) there exists \( n_\omega \) with

\[ -W_{\theta^{n+1}}(\omega) \leq 2h(\theta^{n+1}), \quad n \geq n_\omega. \]

Note that for \( n \geq 2/\log \theta^{-1} \)

\[ \log \theta^{-(n+1)} = (n + 1)\log \theta^{-1} \leq 2n\log \theta^{-1} \leq n^2(\log \theta^{-1})^2 = (\log \theta^{-n})^2. \]

Hence

\[ \log \log \theta^{-(n+1)} \leq \log(\log \theta^{-n})^2 \leq 2\log \log \theta^{-n}. \]

and so we find that

\[ h(\theta^{n+1}) \leq \sqrt{2\theta^{(n+1)/2}}\sqrt{2\log \log \theta^{-n}} \leq 2\sqrt{\theta}h(\theta^n). \]

Combining this with the preceding inequality yields for \( \omega \in \Omega_0 \cap \Omega_\theta \) that

\[ W_{\theta^n}(\omega) \geq (1 - \sqrt{\theta})h(\theta^n) - 2h(\theta^{n+1}) \geq h(\theta^n)(1 - 5\sqrt{\theta}), \]

for infinitely many \( n \). Hence

\[ \limsup_{t \to 0} \frac{W_t(\omega)}{h(t)} \geq 1 - 5\sqrt{\theta}, \quad \omega \in \Omega_0 \cap \Omega_\theta. \]

Finally put \( \Omega^* = \Omega_0 \cap \bigcap_{k \geq 1} \Omega_{1/k} \). Clearly \( P\{\Omega^*\} = 1 \) and \( \limsup_{t \to 0} \frac{W_t(\omega)}{h(t)} \geq \lim_{k \to \infty} (1 - 5/\sqrt{\theta}) = 1 \). QED

As a corollary we have the following result regarding the zero set of the BM that was considered in Exercise 1.28 of Chapter 1.

**Corollary 2.4.6** The point 0 is an accumulation point of the zero set of the BM, i.e. for every \( \varepsilon > 0 \), the BM visits 0 infinitely often in the time interval \([0, \varepsilon]\).

**Proof.** By the law of the iterated logarithm, there exist sequences \( t_n \) and \( s_n \) converging monotonically to 0, such that

\[ \frac{W_{t_n}}{\sqrt{2t_n \log \log 1/t_n}} \to 1, \quad \frac{W_{s_n}}{\sqrt{2s_n \log \log 1/s_n}} \to -1, \quad n \to \infty. \]

The corollary follows from the continuity of Brownian motion paths. QED
2.4.4 Distribution of hitting times

Let \( W \) be a standard Brownian motion and, for \( a > 0 \), let \( \tau_a \) be the (a.s. finite) hitting time of level \( a \) (cf. Example 1.6.9).

**Theorem 2.4.7** For \( a > 0 \) the Laplace transform of the hitting time \( \tau_a \) is given by

\[
E e^{-\lambda \tau_a} = e^{-a \sqrt{2\lambda}}, \quad \lambda \geq 0.
\]

**Proof.** For \( b \geq 0 \), consider the exponential martingales \( M_t = \exp(bW_t - b^2 t/2) \) (see Example 2.1.4). The stopped process \( M_{\tau_a} \) is again a martingale (see Corollary 2.3.14) and is bounded by \( \exp(ab) \). A bounded martingale is uniformly integrable. Hence, by the optional stopping theorem

\[
E M_{\tau_a} = E M_{\tau_a}^\infty = E M_0^\tau_a = E M_0 = 1.
\]

Since \( W_{\tau_a} = a \), it follows that

\[
E e^{b a - b^2 \tau_a/2} = 1.
\]

The expression for the Laplace transform now follows by substituting \( b^2 = 2\lambda \).

We will later see that \( \tau_a \) has the density

\[
x \rightarrow \frac{ae^{-a^2/2x}}{\sqrt{2\pi x^3}}1_{\{x \geq 0\}}.
\]

This can be shown by inverting the Laplace transform of \( \tau_a \).

A formula for the inversion of Laplace transforms are given in BN§4.

We will however use an alternative method in the next chapter. At this point we only prove that although the hitting times \( \tau_a \) are a.s. finite, we have \( E \tau_a = \infty \) for every \( a > 0 \). A process with this property is called null recurrent.

**Corollary 2.4.8** For every \( a > 0 \) it holds that \( E \tau_a = \infty \).

**Proof.** Denote the distribution function of \( \tau_a \) by \( F \). By integration by parts we have for every \( \lambda > 0 \)

\[
E e^{-\lambda \tau_a} = \int_0^\infty e^{-\lambda x} dF(x) = e^{-\lambda x} F(x) \bigg|_0^\infty - \int_0^\infty F(x) d(e^{-\lambda x}) = -\int_0^\infty F(x) d(e^{-\lambda x}).
\]

Combination with the fact that

\[
-1 = \int_0^\infty d(e^{-\lambda x})
\]

it follows that

\[
\frac{1 - E e^{-\lambda \tau_a}}{\lambda} = -\frac{1}{\lambda} \int_0^\infty (1 - F(x)) d(e^{-\lambda x}) = \int_0^\infty (1 - F(x)) e^{-\lambda x} dx.
\]

Now suppose that \( E \tau_a < \infty \). Then by dominated convergence the right-hand side converges to \( E \tau_a \) as \( \lambda \to 0 \). In particular

\[
\lim_{\lambda \to 0} \frac{1 - E e^{-\lambda \tau_a}}{\lambda}
\]

is finite. However, the preceding theorem shows that this is not the case.

QED
2.5 Poisson process and the PASTA property

Let \( N \) be a right continuous Poisson process with parameter \( \mu \) on \((\Omega, \mathcal{F}, P)\). Here the space \( \Omega \) are right-continuous, non-decreasing integer valued paths, such that for each path \( \omega \) one has \( \omega_0 = 0 \) as well as \( \omega_t = \lim_{s \uparrow t} \omega_s + 1 \) for all \( t > 0 \) (cf. construction in Chapter 1.1). The path properties imply all paths in \( \Omega \) to have at most finitely many discontinuities in each bounded time interval. The \( \sigma \)-algebra \( \mathcal{F} \) is the associated \( \sigma \)-algebra that makes the projections on the \( t \)-coordinate measurable.

As we have seen in Example 2.1.5, \( \{N_t - \mu t\}_t \) is a martingale. This implies that \( N_t \) has a decomposition as the sum of a martingale and an increasing process, called Doob-Meyer decomposition.

**Lemma 2.5.1**

\[
\frac{N_t}{t} \xrightarrow{a.s.} \mu, \quad t \to \infty.
\]

**Proof.** See Exercise 2.34. QED

The structure of Poisson paths implies that each path can be viewed as to represent the ‘distribution function’ of a counting measure, that gives measure 1 to each point where \( N(\omega) \) has a discontinuity. In fact, the measure \( \nu_\omega \), with \( \nu_\omega([0, t]) = N_t(\omega) \) is a Lebesgue-Stieltjes measure generated by the trajectory \( N(\omega) \), and \( \nu_\omega(A) \) ‘counts’ the number of jumps of \( N(\omega) \) occurring in \( A \in \mathcal{B}[0, \infty) \).

Denote these successive jumps by \( S_n(\omega), n = 1, \ldots, \) Let \( f : [0, t] \to \mathbb{R} \) be any bounded or non-negative measurable function. Then for each \( \omega \in \Omega, t \geq 0 \), we define

\[
\int_0^t f(s) dN_s(\omega) = \sum_{n=0}^{N_t(\omega)} f(S_n(\omega)) = \sum_{n=0}^{\infty} f(S_n(\omega)). \tag{2.5.1}
\]

This can be derived using the ‘standard machinery’.

**The PASTA property** Let us now consider some \((E, \mathcal{E})\)-valued stochastic process \( X \) on \((\Omega, \mathcal{F}, (\mathcal{F}_t)_t, P)\), where \((E, \mathcal{E})\) is some measure space. Let \( B \in \mathcal{E} \).

The aim is to compare the fraction of time that \( X \)-process spends in set \( B \), with the fraction of time points generated by the Poisson process that the \( X \)-process is in \( B \). We need to introduce some notation: \( U_t = 1_{\{X_t \in B\}} \), and, as usual, \( \lambda \) stands for the Lebesgue measure on \((\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))\).

**Assumption A** \( U \) has ladcag paths, i.o.w. all paths of \( U \) are left continuous and have right limits.

We further define

\[
\begin{align*}
U_t &= \frac{1}{t} \int_0^t U_s d\lambda(s) \\
A_t &= \int_0^t U_s dN_s \\
\tilde{A}_t &= \frac{A_t}{N_t} 1_{\{N_t > 0\}} \\
\mathcal{F}_t &= \sigma(U_s, s \leq t, N_s, s \leq t).
\end{align*}
\]
Here $U_t$ stands for the fraction of time during $(0, t]$ that $X$ spends in set $B$; $A_t$ is the amount of Poisson time points before $t$ at which $X$ is in set $B$, and $\bar{A}_t$ is the fraction of Poisson time points up to time $t$ at which $X$ is in $B$.

**Assumption B Lack of anticipation property** $\sigma(N_{t+u} - N_t, u \geq 0)$ and $F_t$ are independent for all $t \geq 0$.

**Theorem 2.5.2** Under assumptions A and B, there exists a finite random variable $U_1$ such that $U_t \overset{a.s.}{\to} U_1$, there exists a random variable $\bar{A}_1$ such that $\bar{A}_t \overset{a.s.}{\to} \bar{A}_1$ and then $\bar{A}_1 = U_1$.

The proof requires a number of steps.

**Lemma 2.5.3** Suppose that assumptions A and B hold. Then $E A_t = \lambda t E U_t = \lambda E \int_0^t U_s d\lambda(s)$.

**Proof.** In view of (2.5.1) and the sample properties of $N$, this implies that we can approximate $A_t$ by

$$A_{n,t} = \sum_{k=0}^{n-1} U_{k+1} \frac{N_{k+1} - N_k}{n}.$$  

In other words, $A_{n,t} \overset{a.s.}{\to} A_t$. Now, evidently $0 \leq A_{n,t} \leq N_t$. Since $E|N_t| = EN_t < \infty$, we can apply the dominated convergence theorem and Assumption B to obtain that

$$E A_t = \lim_{n \to \infty} E A_{n,t} = \lim_{n \to \infty} \frac{\mu t}{n} \sum_{k=0}^{n-1} E U_{k+1}.$$  

Similarly, one can derive that

$$\frac{\mu t}{n} \sum_{k=0}^{n-1} E U_{k+1} \overset{a.s.}{\to} \mu \int_0^t U_s d\lambda(s), \quad n \to \infty.$$  

Using that

$$0 \leq \frac{\mu t}{n} \sum_{k=0}^{n-1} E U_{k+1} \leq \mu t,$$

we can apply the dominated convergence theorem to obtain, that

$$E \lim_{n \to \infty} \frac{\mu t}{n} \sum_{k=0}^{n-1} U_{k+1} = \mu E \int_0^t U_s d\lambda(s).$$  

Combining (2.5.2) and (2.5.3) yields

$$E A_t = \mu E \int_0^t U_s d\lambda(s).$$

QED
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**Corollary 2.5.4** Suppose that assumptions A and B hold. \( \mathbb{E}(A_t - A_s \mid \mathcal{F}_s) = \mu \mathbb{E}(\int_s^t U_v d\lambda(v) \mid \mathcal{F}_s) \) a.s.

**Proof.** The lemma implies that \( \mathbb{E}(A_t - A_s) = \mu \mathbb{E}(\int_s^t U_v d\lambda(v)) \). Define

\[
A_{n,s,t} = \sum_{ns/t \leq k \leq n-1} \frac{U_{nk}}{n} [N_{(k+1)t/n} - N_{kt/n}].
\]

Then, analogously to the above proof, \( A_{n,s,t} \xrightarrow{a.s.} A_t - A_s = \int_s^t U_v d\lambda(v) \).

We use conditional dominated convergence (BN Theorem 7.2 (vii)). This implies that \( \mathbb{E}(A_{n,s,t} \mid \mathcal{F}_s) \rightarrow \mathbb{E}(A_t - A_s \mid \mathcal{F}_s) \) a.s. On the other hand

\[
\mathbb{E}(A_{n,s,t} \mid \mathcal{F}_s) = \mathbb{E}(\frac{\mu}{n} \sum_{ns/t \leq k \leq n-1} \frac{U_{nk}}{n} \mid \mathcal{F}_s).
\]

By another application of conditional dominated convergence, using boundedness of the function involved, the right-hand side converges a.s. to \( \mathbb{E}(\mu \int_s^t U_v d\lambda \mid \mathcal{F}_s) \). QED

Next define \( R_t = A_t - \mu U_t \). By virtue of Corollary 2.5.4 \( \{R_t\}_t \) is an \( (\mathcal{F}_t)_t \)-adapted martingale.

**Lemma 2.5.5** Suppose that assumptions A and B hold. Then \( R_t/t \xrightarrow{a.s.} 0 \).

**Proof.** Note that \( \{R_{nh}\}_n \) is an \( (\mathcal{F}_{nh})_n \)-adapted discrete time martingale for any \( h > 0 \). By virtue of Theorem 2.2.22

\[
\frac{R_{nh}}{n} \rightarrow 0,
\]

on the set

\[
A = \{ \sum_{k=1}^{\infty} \frac{1}{k^2} \mathbb{E}((R_{nk+1} - R_{kh})^2 \mid \mathcal{F}_{kh}) < \infty \}.
\]

Note that

\[
|R_t - R_s| \leq N_t - N_s + \mu(t - s).
\]

It follows that

\[
\mathbb{E}(R_t - R_s)^2 \leq \mathbb{E}(N_t - N_s)^2 + 3\mu^2(t - s)^2 = 4\mu^2(t - s)^2 + \mathbb{E}(R_t - R_s).
\]

Consider the random variables \( Y_n = \sum_{k=1}^{\infty} \frac{1}{k^2} \mathbb{E}((R_{nk+1} - R_{kh})^2 \mid \mathcal{F}_{kh}) \), \( n = 1, \ldots \). By (2.5.5),

\[
\mathbb{E}|Y_n| = \mathbb{E}Y_n \leq \sum_{k=1}^{\infty} (4\mu^2h^2 + \mu h) \frac{1}{k^2} < \infty,
\]

for all \( n \), hence \( \{Y_n\}_n \) is bounded in \( L_1 \). \( Y_n \) is an increasing sequence that converges to a limit \( Y_\infty \), that is possibly not finite everywhere. By monotone convergence and \( L_1 \)-boundedness \( \mathbb{E}Y_n \rightarrow \mathbb{E}Y_\infty < \infty \). As a consequence \( Y_\infty \) must be a.s. finite. In other words, \( \mathbb{P}\{A\} = 1 \). That is \( R(nh)/nh \rightarrow 0 \).

Let \( \Omega^* \) be the intersection of \( A \) and the set where \( N_t/t \rightarrow \mu \). By Lemma 2.5.1 \( \mathbb{P}\{\Omega^*\} = 1 \).

The lemma is proved if we show that \( R_t/t \rightarrow 0 \) on the set \( \Omega^* \). Let \( \omega \in \Omega^* \). Fix \( t \) and let \( n_t \) be such that \( t \in [n_th, (n_t + 1)h) \). By virtue of (2.5.4)

\[
|R_t(\omega) - R_{nh}(\omega)| \leq N_t(\omega) - N_{nh}(\omega) + \mu h.
\]
By another application of Lemma 2.5.1
\[
\frac{R_t(\omega)}{t} \leq \frac{n_{th}}{t} \cdot \frac{R_{n_{th}}(\omega) + N(t, \omega) - N_{n_{th}}(\omega) + \mu h}{n_{th}} \to 0, \to \infty.
\]
QED

Now we can finish the proof of the theorem. It follows from the relation
\[
\frac{R_t}{t} = \frac{A_t}{N_t} \frac{N_{t^1} \{N_t > 0\}}{t} - \mu \bar{U}_t,
\]
whilst noting that \((N_t/t)1_{\{N_t > 0\}} \to \mu \text{ on } \Omega^*\).
2.6 Exercises

Discrete-time martingales

Exercise 2.1 Prove the assertion in Example 2.1.2.

Exercise 2.2 Prove the assertion in Example 2.1.3.

Exercise 2.3 Show that the processes defined in Example 2.1.4 are indeed martingales.

Exercise 2.4 (Kolmogorov 0-1 Law) Let $X_1, X_2, \ldots$ be i.i.d. random variables and consider the tail $\sigma$-algebra defined by

$$\mathcal{T} = \bigcap_{n=1}^{\infty} \sigma(X_n, X_{n-1}, \ldots).$$

a) Show that for every $n$, $\mathcal{T}$ is independent of the $\sigma$-algebra $\sigma(X_1, \ldots, X_n)$ and conclude that for every $A \in \mathcal{T}$

$$P\{A\} = E(1_A | X_1, \ldots, X_n), \quad \text{a.s.}$$

b) Give a “martingale proof” of Kolmogorov’s 0–1 law: for every $A \in \mathcal{T}$, $P\{A\} = 0$ or $P\{A\} = 1$.

c) Give an example of an event $A \in \mathcal{T}$.

Exercise 2.5 (Law of large Numbers) In this exercise we present a “martingale proof” of the law of large numbers. Let $X_1, X_2, \ldots$ be random variables with $E|X_1| < \infty$. Define $S_n = \sum_{i=1}^{n} X_i$ and $\mathcal{F}_n = \sigma(S_n, S_{n+1}, \ldots)$.

a) Note that for $i = 1, \ldots, n$, the distribution of the pair $(X_i, S_n)$ is independent of $i$. From this fact, deduce that $E(X_n | \mathcal{F}_n) = S_n/n$, and that consequently

$$E\left(\frac{1}{n}S_n | \mathcal{F}_n\right) = \frac{1}{n+1}S_{n+1}, \quad \text{a.s.}$$

ii) Show that $S_n/n$ converges almost surely to a finite limit.

iv) Derive from Kolmogorov’s 0–1 law that the limit must be a constant and determine its value.

Exercise 2.6 Consider the proof of Theorem 2.2.18. Prove that for the stopping time $\tau$ and the event $A \in \mathcal{F}_\tau$ it holds that $A \cap \{\tau < \infty\} \in \mathcal{G}$.

Exercise 2.7 Let $X, Y$ be two integrable random variables defined on the same space $\Omega$. Let $\mathcal{F}$ be a $\sigma$-algebra on $\Omega$.

a) Suppose that $X, Y$ are both $\mathcal{F}$-measurable. Show that $X \geq Y$ a.s. if and only if $E1_{\{A\}}X \geq E1_{\{A\}}Y$ for all $A \in \mathcal{F}$.

b) Suppose that $Y$ is $\mathcal{F}$-measurable. Show that $E(X | \mathcal{F}) \leq Y$ a.s. together with $EX = EY$ implies $E(X | \mathcal{F}) = Y$ a.s.
Exercise 2.8 Let $M$ be a martingale such that $\{M_{n+1} - M_n\}_{n \geq 1}$ is a bounded process. Let $Y$ be a bounded predictable process. Let $X = Y \cdot M$. Show that $E X_\tau = 0$ for $\tau$ a finite stopping time with $E \tau < \infty$.

Exercise 2.9 Let $X_1, X_2, \ldots$ be an i.i.d. sequence of Bernoulli random variables with probability of success equal to $p$. Put $F_n = \sigma(X_1, \ldots, X_n)$, $n \geq 1$. Let $M$ be a martingale adapted to the generated filtration. Show that the Martingale Representation Property holds: there exists a constant $m$ and a predictable process $Y$ such that $M_n = m + (Y \cdot S)_n$, $n \geq 1$, where $S_n = \sum_{k=1}^n (X_k - p)$.

Exercise 2.10 Let $X_1, \ldots$ be a sequence of independent random variables with $\sigma_n^2 = EX_n^2 < \infty$ and $EX_n = 0$ for all $n \geq 1$. Consider the filtration generated by $X$ and define the martingale $M$ by $M_n = \sum_{i=1}^n X_i$. Determine $\langle M \rangle$.

Exercise 2.11 Let $M$ be a martingale with $EM_n^2 < \infty$ for every $n$. Let $C$ be a bounded predictable process and define $X = C \cdot M$. Show that $EX_n^2 < \infty$ for every $n$ and that $\langle X \rangle = C^2 \cdot \langle M \rangle$.

Exercise 2.12 Let $M$ be a martingale with $EM_n^2 < \infty$ for every $n$ and let $\tau$ be a stopping time. We know that the stopped process is a martingale as well. Show that $E(M_n^\tau)^2 < \infty$ for all $n$ and that $\langle M^\tau \rangle_n = \langle M \rangle_{n \wedge \tau}$.

Exercise 2.13 Let $(C_n)_n$ be a predictable sequence of random variables with $EC_n^2 < \infty$ for all $n$. Let $(\epsilon_n)_n$ be a sequence with $E\epsilon_n = 0$, $E\epsilon_n^2 = 1$ and $\epsilon_n$ independent of $F_{n-1}$ for all $n$. Let $M_n = \sum_{i \leq n} C_i \epsilon_i$, $n \geq 0$. Compute the conditional variance process $A$ of $M$. Take $p > 1/2$ and consider $N_n = \sum_{i \leq n} C_i \epsilon_i / (1 + A_i)^p$. Show that there exists a random variable $N_\infty$ such that $N_n \to N_\infty$ a.s. Show (use Kronecker’s lemma) that $M_n / (1 + A_n)^p$ has an a.s. finite limit.

Exercise 2.14 i) Show that the following generalisation of the optional stopping Theorem 2.2.18 holds. Let $M$ be a uniformly integrable supermartingale. Then the family of random variables $\{M_\tau \mid \tau \text{ is a finite stopping time} \}$ is UI and $E(M_\tau \mid F_\sigma) \leq M_\sigma$, a.s. for stopping times $\sigma \leq \tau$. Hint: use Doob decomposition.

ii) Give an example of a non-negative martingale for which $\{M_\tau \mid \tau \text{ stopping time} \}$ is not UI.

Exercise 2.14* Show for a non-negative supermartingale $M$ that for all $\lambda > 0$

$$\lambda P\{\sup_n M_n \geq \lambda\} \leq E(M_0).$$

Exercise 2.15 Consider the unit interval $I = [0, 1]$ equipped with the Borel-$\sigma$-algebra $\mathcal{B}([0, 1])$ and the Lebesgue measure. Let $f$ be an integrable function on $I$. Let for $n = 1, 2, \ldots$

$$f_n(x) = 2^n \int_{(k-1)2^{-n}}^{k2^{-n}} f(y) dy, \quad (k - 1)2^{-n} \leq x < k2^{-n},$$

and define $f_n(1) = 1$ (the value $f_n(1)$ is not important). Finally, we define $\mathcal{F}_n$ as the $\sigma$-algebra generated by intervals of the form $[(k - 1)2^{-n}, k2^{-n})$, $1 \leq k \leq 2^n$. 


2.6. EXERCISES

i) Argue that $\mathcal{F}_n$ is an increasing sequence of $\sigma$-algebras.

ii) Show that $(f_n)_n$ is a martingale.

iii) Use Lévy’s Upward Theorem to prove that $f_n \to f$, a.s. and in $L_1$, as $n \to \infty$.

Exercise 2.16 (Martingale formulation of Bellman’s optimality principle) Suppose your winning per unit stake on game $n$ are $\epsilon_n$, where the $\epsilon_n$ are i.i.d. r.v.s with

$$P(\epsilon_n = 1) = p = 1 - P(\epsilon_n = -1),$$

with $p > 1/2$. Your bet $\alpha_n$ on game $n$ must lie between 0 and $Z_{n-1}$, your capital at time $n - 1$. Your object is to maximise your ‘interest rate’ $E \log(Z_N/Z_0)$, where $N = \text{length of the game}$ is finite and $Z_0$ is a given constant. Let $\mathcal{F}_n = \sigma(\epsilon_1, \ldots, \epsilon_n)$ be your ‘history’ up to time $n$. Let $\{\alpha_n\}_n$ be an admissible strategy, i.o.w. a predictable sequence. Show that $\log(Z_n) - n\alpha$ is a supermartingale with $\alpha$ the entropy given by

$$\alpha = p \log p + (1-p) \log(1-p) + \log 2.$$ 

Hence $E \log(Z_n/Z_0) \leq n\alpha$. Show also that for some strategy $\log(Z_n) - n\alpha$ is a martingale. What is the best strategy?

Exercise 2.17 Consider a monkey typing one of the numbers 0, 1, \ldots, 9 at random at each of times 1, 2, \ldots, $U_i$ denotes the $i$-th number that the monkey types. The sequence of numbers $U_1, U_2, \ldots$, form an i.i.d. sequence uniformly drawn from the 10 possible numbers.

We would like to know how long it takes till the first time $T$ that the monkey types the sequence 1231231231. More formally,

$$T = \min \{n \mid n \geq 10, U_{n-9}U_{n-8} \cdots U_n = 1231231231\}.$$ 

First we need to check that $T$ is an a.s. finite, integrable r.v. There are many ways to do this.

a) Show that $T$ is an a.s. finite, integrable r.v. A possibility for showing this, is to first show that the number of consecutive 10-number words typed till the first occurrence of 1231231231 is a.s. finite, with finite expectation.

In order to actually compute $E T$, we will associate a gambling problem with it.

Just before each time $t = 1, 2, 3, \ldots$, a new gambler arrives into the scene, carrying €1 in his pocket. He bets €1 that the next number (i.e. the $t$-th number) will be 1. If he loses, he leaves; if he wins his receives 10 times his bet, and so he will have a total capital of €10. He next bets all of his capital on the event that the $(t + 1)$-th number will be 2. If he loses, he leaves; if he wins, he will have a capital of €10$^2$. This is repeated throughout the sequence 1231231231. So, if the gambler wins the second time, his third bet is on the number 3, and so on, till the moment that either he loses, or the monkey has typed the desired sequence. Note that any gambler entering the game after the monkey typed the desired sequence, cannot play anymore, he merely keeps his initial capital intact.

b) Define a martingale $\{X_n - 1\}_{n=0, \ldots}$, such that $X_n$ is the total capital of the first gambler after his $n$-th game and hence $X_n - 1$ his total gain. Similarly associate with the $k$-th gambler (who enters the game at time $k$) a martingale $X_n - 1$, where $X_n$ is his capital after the $n$-th number that the monkey typed. Write $M_n = \sum_{k=1}^n X_n^k$. Then argue that $(M_n - n)_n$ is a bounded martingale associated with the total gain of all gamblers that entered the game at time $n$ latest.
Exercise 2.18 Let \( X_i, 0 = 1, 2, \ldots \), be independent, integer valued random variables, with \( X_1, \ldots \) identically distributed. Assume that \( \mathbb{E}|X_1|, \mathbb{E}X_1^2 < \infty \). Let \( a < b, a, b \in \mathbb{Z} \), and assume that \( X_i \neq 0 \). Consider the stochastic process \( S = (S_n)_{n \in \mathbb{N}} \), with \( S_n = \sum_{i=0}^n X_i \) the \((n + 1)-th\) partial sum.

We desire to show the intuitively clear assertion that the process leaves \((a, b)\) in finite expected time, given that it starts in \((a, b)\). Define \( \tau_{a,b} = \min\{n \mid S_n \notin (a, b)\} \) and

\[
f(x) = \mathbb{P}\{S_i \in (a, b), i = 0, 1, 2, \ldots \mid S_0 = x\}.
\]

Note that \( f(x) = 0 \) whenever \( x \notin (a, b)! \) Let now \( S_0 \equiv x_0 \in (a, b) \).

a) Show that \( (f(S_{n \wedge \tau_{a,b}}))_n \) is a martingale.

b) Show that this implies that \( \tau_{a,b} \) is a.s. finite. Hint: consider the maximum of \( f \) on \((a, b)\) and suppose that it is strictly positive. Derive a contradiction.

Fix \( x_0 \in (a, b) \), let \( S_0 \equiv x_0 \), and assume that \( X_1 \) is bounded.

c) Show that \( \tau_{a,b} \) is an a.s. finite and integrable r.v. Hint: you may consider the processes \((S_n - n\mathbb{E}X_1)_{n \in \mathbb{N}}\), and, if \( \mathbb{E}X_1 = 0, (S_n^2 - n\mathbb{E}X_1^2)_{n \in \mathbb{N}}\).

d) Show that \( \tau_{a,b} \) is a.s. finite and integrable also if \( X_1 \) is not bounded.

e) Derive an expression for \( \mathbb{E}\tau_{a,b} \) (in terms of \( x_0 \), \( a \) and \( b \)) in the special case that \( \mathbb{P}\{X_1 = 1\} = \mathbb{P}\{X_1 = -1\} = 1/2 \).

f) Now assume that \( \mathbb{E}X_1 \leq 0 \). Let \( \tau_a = \min\{n \geq 0 \mid S_n \leq a\} \). Show that \( \tau_a \) is a.s. finite. Hint: consider \( \tau_{a,n} \) and let \( n \) tend to \( \infty \).

Exercise 2.18’ Another approach of the first part of Exercise 2.18. Let \( X_i, i = 0, \ldots, \) all be defined on the same underlying probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Let

\[
f = 1_{\{S_i \in (a,b), i = 0, \ldots\}}.
\]

a’) The stochastic process \( M_n = \mathbb{E}(f \mid S_0, \ldots, S_n) \) is a martingale that converges a.s. and in \( L^1 \) to \( M_\infty = \mathbb{E}\{f \mid S_0, \ldots\} \). Argue that

\[
\mathbb{E}\{f \mid S_0 = x_0, \ldots, S_n = x_n\} = \mathbb{E}\{f \mid S_0 = x_n\},
\]

for all \( x_0, \ldots, x_{n-1} \in (a, b) \).

Use this to show for all \( x \in (a, b) \) that

\[
a_x = \sum_y \mathbb{P}\{X_1 = y\}a_{x+y},
\]

where \( a_x = \mathbb{E}\{f \mid S_0 = x\} \) (note: \( a_x \) is a real number!).

b’) Show that this implies that \( a_x = 0 \) for all \( x \in (a, b) \). Hint: consider the point \( x^* = \{x \in (a, b) \mid a_x = \max_{y \in (a, b)} a_y\} \). Let now \( S_0 = x_0 \in (a, b) \) be given. Conclude from the previous that \( \tau_{a,b} \) is a.s. finite.
Exercise 2.19 (Galton-Watson process) This is a simple model for population growth, growth of the number of cells, etc.

A population of cells evolves as follows. In every time step, every cell splits into 2 cells with probability \( p \) or it dies with probability \( 1 - p \), independently of the other cells and of the population history. Let \( N_t \) denote the number of cells at time \( t \), \( t = 0, 1, 2, \ldots \). Initially, there is only 1 cell, i.e. \( N_0 = 1 \).

We can describe this model formally by defining \( Z^n_t \), \( n = 1, \ldots, N_t \), \( t = 0, 1, 2, \ldots \) to be i.i.d. random variables with
\[
\Pr\{Z^n_t = 2\} = p = 1 - \Pr\{Z^n_t = 0\},
\]
and then \( N_{t+1} = \sum_{n=1}^{N_t} Z^n_t \). Let \( \{F_t\}_{t=0,1,\ldots} \) be the natural filtration generated by \( \{N_t\}_{t=0,1,\ldots} \).

i) Argue or prove that
\[
\Pr\{N_{t+1} = 2 \mid F_t\} = \mathbb{E}\{N_{t+1} = 2 \mid N_t\} = \binom{N_t}{y} p^y (1 - p)^{N_t-y}.
\]
Hence, conditional on \( F_t \), \( N_{t+1}/2 \) has a binomial distribution with parameters \( N_t \) and \( p \).

ii) Let \( \mu = \mathbb{E}\{N_1\} \). Show that \( N_t/\mu^t \) is a martingale with respect to \( \{F_t\} \), bounded in \( L^1 \).

iii) Assume that \( \mu < 1 \). Show that \( \mathbb{E}N_t = \mu^t \) and that the population dies out a.s. in the long run.

iv) Assume again that \( \mu < 1 \). Show that \( M_t = \alpha^N_{t}1_{\{N_t>0\}} \) is a contracting supermartingale for some \( \alpha > 1 \), i.e. there exist \( \alpha > 1 \) and \( 0 < \beta < 1 \) such that
\[
\mathbb{E}(M_{t+1} \mid F_t) \leq \beta M_t, \quad t = 1, 2, \ldots.
\]

v) Show that this implies that \( \mathbb{E}(T) < \infty \) with \( T = \min\{t \geq 1 \mid N_t = 0\} \) the extinction time.

Exercise 2.20 (Continuation of Exercise 2.19) From now on, assume the critical case \( \mu = 1 \), and so \( N_t \) is itself a martingale. Define \( \tau_{0,N} = \min\{t \mid N_t = 0 \text{ or } N_t \geq N\} \). Further define
\[
M_t = N_t \cdot 1_{\{N_t \in \{1,\ldots,N-1\}\}}.
\]

i) Argue that \( M_t \) is a supermartingale and that there exists a constant \( \alpha < 1 \) (depending on \( N \)) such that
\[
\mathbb{E}M_{t+N} \leq \alpha \mathbb{E}N_t.
\]
Show that this implies that \( \Pr\{\tau_{0,N} = \infty\} = 0 \).

ii) Show that \( \Pr\{N_{\tau_{0,N}} \geq N\} \leq 1/N \). Show that this implies the population to die out with probability 1.

iii) Is \( \{N_t\}_t \) u.i. in the case of \( \mu < 1 \)? And if \( \mu = 1 \)?
Exercise 2.21 Let $X_i, i = 0, \ldots$ be independent $\mathbb{Z}$-valued random variables, with $E|X_i| < \infty$. We assume that $X_1, X_2, \ldots$ are identically distributed with values in $\{-1, 0, 1\}$, that all have positive probability. Then $S_n = \sum_{i=0}^n X_i$ is a discrete time Markov chain taking values in $\mathbb{Z}$. Suppose that $X_0$ has distribution $\nu = \delta_x$, with $x \in (a, b) \subset \mathbb{Z}_+$. Define $\tau_y = \min\{n > 0 : S_n = y\}$ and let $\tau = \tau_a \wedge \tau_b$. We want to compute $P\{\tau_b < \tau_a\}$ and $E\{\tau\}$ (recall that $S_0 \equiv x!$). Let first $EX_1 \neq 0$.

i) Show that $\tau$ is a stopping time w.r.t. a suitable filtration. Show that $\tau$ is finite a.s. Hint: use the law of large numbers.

ii) We want to define a function $f : \mathbb{Z} \to \mathbb{R}$, such that $\{f(S_n)\}_n$ is a discrete-time martingale. It turns out that we can take $f(z) = e^{\alpha z}$, $z \in \mathbb{R}$, for suitably chosen $\alpha$.

Show that there exists $\alpha \neq 0$, such that $e^{\alpha S_n}$ is a martingale. Use this martingale to show that

$$P\{\tau_b < \tau_a\} = \frac{e^{\alpha x} - e^{\alpha a}}{e^{\alpha b} - e^{\alpha a}}.$$

If $P\{X_1 = 1\} = p = 1 - P\{X_1 = -1\}$ (that is: $X_i$ takes only values $\pm 1$), then

$$e^{\alpha} = \frac{1 - p}{p} \quad \text{or} \quad \alpha = \log(1 - p) - \log p.$$

Show this.

iii) Show that $S_n - nEX_1$ is martingale. Show that

$$E\{\tau\} = \frac{(e^{\alpha x} - e^{\alpha b})(x - a) + (e^{\alpha x} - e^{\alpha a})(b - x)}{(e^{\alpha b} - e^{\alpha a})EX_1}.$$

iv) Let now $EX_1 = 0$. Show that $\tau < \infty$ a.s. (hint: use the Central Limit Theorem). Show for $x \in (a, b)$ that

$$P\{\tau_a < \tau_b\} = \frac{b - x}{b - a}$$

and

$$E\{\tau\} = \frac{(x - a)(b - x)}{EX_1^2},$$

by constructing suitable martingales.

Continuous-time martingales

Exercise 2.22 Prove Corollary 2.3.3. Hint: we know from Theorem 2.3.2 that the left limits exist for all $t$ on an $\mathcal{F}_\infty$-measurable subset $\Omega'$ of probability 1, along rational sequences. You now have to consider arbitrary sequences.

Exercise 2.23 Show that the process constructed in the proof of Theorem 2.3.5 is cadlag and a supermartingale, if $M$ is a supermartingale.

Exercise 2.24 Prove the ‘only if’ part of Theorem 2.3.5.
2.6. EXERCISES

Exercise 2.25 Prove Theorem 2.3.9 from LN. You may use Theorem 2.3.7.

Exercise 2.26 Show that for every \( a \neq 0 \), the exponential martingale of Example 2.1.4 converges to 0 a.s. as \( t \to \infty \). (Hint: use for instance the recurrence of Brownian motion) Conclude that these martingales are not uniformly integrable.

Exercise 2.27 Give an example of two stopping times \( \sigma \leq \tau \) and a martingale \( M \) that is bounded in \( L^1 \) but not uniformly integrable, for which the equality \( E(M_\tau | F_\sigma) = M_\sigma \) a.s. fails. (Hint: see Exercise 2.26).

Exercise 2.28 Let \( M \) be a positive, continuous martingale that converges a.s. to zero as \( t \) tends to infinity.

a) Prove that for every \( x > 0 \)
\[
P\{\sup_{t \geq 0} M_t > x | F_0\} = 1 \wedge \frac{M_0}{x} \quad \text{a.s.}
\]
(Hint: stop the martingale when it gets to above the level \( x \)).

b) Let \( W \) be a standard BM. Using the exponential martingales of Example 2.1.4, show that for every \( a > 0 \) the random variable
\[
\sup_{t \geq 0} (W_t - \frac{1}{2} at)
\]
has an exponential distribution with parameter \( a \).

Exercise 2.29 Let \( W \) be a BM and for \( a \in \mathbb{R} \) let \( \tau_a \) be the first time that \( W \) hits \( a \). Suppose that \( a > 0 > b \). By considering the stopped martingale \( W_{\tau_a \wedge \tau_b} \), show that
\[
P\{\tau_a < \tau_b\} = \frac{-b}{a-b}.
\]

Exercise 2.30 Consider the setup of the preceding exercise. By stopping the martingale \( W_t^2 - t \) at an appropriate stopping time, show that \( E(\tau_a \wedge \tau_b) = -ab \). Deduce that \( E\tau_a = \infty \).

Exercise 2.31 Let \( W \) be a BM and for \( a > 0 \), let \( \tilde{\tau}_a \) be the first time that \( |W| \) hit the level \( a \).

a) Show that for every \( b > 0 \), the process \( M_t = \cosh(b|W_t|) \exp(b^2t/2) \) is a martingale.

b) Find the Laplace transform of the stopping time \( \tilde{\tau}_a \).

c) Calculate \( E\tilde{\tau}_a \).

Exercise 2.32 (Empirical distributions) Let \( X_1, \ldots, X_n \) be i.i.d. random variables, each with the uniform distribution on \([0, 1]\). For \( 0 \leq t < 1 \) define
\[
G_n(t) = \frac{1}{n} \#\{k \leq n \mid X_k \leq t\} = \frac{1}{n} \sum_{k=1}^{n} 1\{X_k \leq t\}.
\]
In words, \( G_n(t) \) is the fraction of \( X_k \) that have value at most \( t \). Denote
\[
\mathcal{F}_n(t) = \sigma(1\{X_1 \leq s\}, \ldots, 1\{X_n \leq s\}, s \leq t)
\]
and \( \mathcal{G}_n(t) = \sigma(G_n(s), s \leq t) \).
Extracted Text

i) Prove that for $0 \leq t \leq u < 1$ we have

$$E\{G_n(u) | \mathcal{F}_n(t)\} = G_n(t) + (1 - G_n(t)) \frac{u - t}{1 - t}.$$ 

Show that this implies that

$$E\{G_n(u) | \mathcal{G}_n(t)\} = G_n(t) + (1 - G_n(t)) \frac{u - t}{1 - t}.$$ 

ii) Show that the stochastic process $M_n = (M_n(t))_{t \in [0,1)}$ defined by

$$M_n(t) = \frac{G_n(t) - t}{1 - t}$$ 

is a continuous-time martingale w.r.t $\mathcal{G}_n(t), t \in [0,1).$

iii) Is $M_n$ UI? Hint: compute $\lim_{t \uparrow 1} M_n(t).$

iv) We extend the process $M_n$ to $[0, \infty)$, by putting $M_n(t) = 1$ and $G_n(t) = \sigma(X_1, \ldots, X_n)$, for $t \geq 1$. Show that $(M_n(t))_{t \in [0,\infty)}$ is an $L^1$-bounded submartingale relative to $(\mathcal{G}_n(t))_{t \in [0,\infty)}$, that converges in $L^1$, but is not UI.

Exercise 2.33 Let $(W_t)_t$ be a standard Brownian motion, and define

$$X_t = W_t + ct,$$

for some constant $c$. The process $X_t$ is called Brownian motion with drift. Fix some $\lambda > 0$.

i) Show that

$$M_t := e^{\theta X_t - \lambda t}$$

is a martingale (with respect to the natural filtration) if and only if $\theta = \sqrt{c^2 + 2\lambda} - c$ or $\theta = -\sqrt{c^2 + 2\lambda} - c$.

Next, let $H_x = \inf\{t > 0 | X_t = x\}$.

ii) Argue for $x \neq 0$ that $H_x$ is a stopping time.

iii) Show that

$$E(e^{-\lambda H_x}) = \begin{cases} e^{-x(\sqrt{c^2 + 2\lambda} - c)}, & x > 0 \\ e^{-x(-\sqrt{c^2 + 2\lambda} - c)}, & x < 0. \end{cases}$$

iv) Use the result from (iii) to prove that for $x > 0$

$$P\{H_x < \infty\} = \begin{cases} 1, & c \geq 0 \\ e^{-2|x|c}, & c < 0. \end{cases}$$

v) Explain why this result is reasonable.

Exercise 2.34 Let $N = \{N(t)\}_{t \geq 0}$ be a Poisson process (see Definition in Ch1.1). Show that $\{N(t) - \lambda t\}_{t \geq 0}$ is a martingale. Then prove Lemma 2.5.1. Hint: use the martingale LLN given in Section 2.2.6.
Chapter 3

Markov Processes

3.1 Basic definitions: a mystification?

Notational issues To motivate the conditions used later on to define a Markov process, we will recall the definition of a discrete-time and discrete-space Markov chain. Let \( E \) be a discrete space, and \( E \) the \( \sigma \)-algebra generated by the one-point sets: \( E = \sigma \{ \{x\} | x \in E \} \). Let \( X = \{X_n\}_{n=0,1,...} \) be an \((E, E)\)-valued stochastic process defined on some underlying probability space \((\Omega, \mathcal{F}, P)\). In Markov chain theory, it is preferred not to fix the distribution of \( X_0 \), i.e. the initial distribution. In our notation we will therefore incorporate the dependence on the initial distribution.

The initial distribution of the process is always denoted by \( \nu \) in these notes. The associated probability law of \( X \) and corresponding expectation operator will be denoted by \( P_\nu \) and \( E_\nu \), to make the dependence on initial distribution visible in the notation. If \( X_0 = x \) a.s. then we write \( \nu = \delta_x \) and use the shorthand notation \( P_x \) and \( E_x \) (instead of \( P_\delta_x \) and \( E_\delta_x \)). \( E \) is called the state space.

Assume hence that \( X \) is a stochastic process on \((\Omega, \mathcal{F}, P_\nu)\). Then \( X \) is called a Markov chain with initial distribution \( \nu \), if there exists an \( E \times E \) stochastic matrix \( P \), such that

i) \( P_\nu \{ X_0 \in B \} = \nu(B) \) for all \( B \in \mathcal{E} \);

ii) The Markov property holds, i.e. for all \( n = 0,1,..., x_0, \ldots, x_n, x_{n+1} \in E \)

\[ P_\nu \{ X_{n+1} = x_{n+1} | X_0 = x_0, \ldots, X_n = x_n \} = P_\nu \{ X_{n+1} = x_{n+1} | X_n = x_n \} = P(x_n, x_{n+1}). \]

Recall that

\[ P_\nu \{ X_{n+1} = x_{n+1} | \sigma(X_n) \} = E_\nu \{ 1_{\{X_{n+1} = x_{n+1}\}} | \sigma(X_n) \} \]

is a function of \( X_n \). In the Markov case this is \( P(X_n, x_{n+1}) \). Then \( P_\nu \{ X_{n+1} = x_{n+1} | X_n = x_n \} = P(x_n, x_{n+1}) \) is simply the evaluation of that function at the point \( X_n = x_n \). These conditional probabilities can be computed by

\[ P_\nu \{ X_{n+1} = x_{n+1} | X_n = x_n \} = \begin{cases} \frac{P_\nu \{ X_{n+1} = x_{n+1}, X_n = x_n \}}{P_\nu \{ X_n = x_n \}}, & \text{if } P_\nu \{ X_n = x_n \} > 0 \\ \text{anything you like} & \text{if } P_\nu \{ X_n = x_n \} = 0 \end{cases} \]

\(^1\)The \( E \times E \) matrix \( P \) is stochastic if it is a non-negative matrix with row sums equal to 1.
We can now rephrase the Markov property as follows: for all \( n \in \mathbb{Z}_+ \) and \( y \in E \)

\[
P_{\nu}\{X_{n+1} = y \mid \mathcal{F}_n^X\} = P_{\nu}\{X_{n+1} = y \mid \sigma(X_n)\} = P(X_n, y), \quad \text{a.s.} \tag{3.1.1}
\]

It is a straightforward computation that

\[
P_{\nu}\{X_{n+m} = y \mid \mathcal{F}_n^X\} = P^m(X_n, y),
\]

is the \((X_n, y)\)-th element of the \(m\)-th power of \(P\). Indeed, for \(m = 2\)

\[
P_{\nu}\{X_{n+2} = y \mid \mathcal{F}_n^X\} = \mathbb{E}_{\nu}(1_{\{y\}}(X_{n+2}) \mid \mathcal{F}_n^X)
= \mathbb{E}_{\nu}(\mathbb{E}_{\nu}(1_{\{y\}}(X_{n+2}) \mid \mathcal{F}_{n+1}^X) \mid \mathcal{F}_n^X)
= \mathbb{E}_{\nu}(P(X_{n+1}, y) \mid \mathcal{F}_n^X)
= \mathbb{E}_{\nu}\left(\sum_{x \in E} 1_{\{x\}}(X_{n+1}) \cdot P(X_{n+1}, y) \mid \mathcal{F}_n^X\right)
= \sum_{x \in E} P(x, y)\mathbb{E}_{\nu}(1_{\{x\}}(X_{n+1}) \mid \mathcal{F}_n^X)
= \sum_{x \in E} P(x, y)P(X_n, x) = P^2(X_n, y).
\]

In steps (3.1.2) and after, we use discreteness of the state space as well linearity of conditional expectations. In fact we have proved a more general version of the Markov property to hold. To formulate it, we need some more notation. But first we will move on to Markov chains on a general measurable space.

**Discrete time Markov chains on a general state space** The one point sets need not be measurable in general space. The notion of a stochastic matrix generalises to the notion of a transition kernel.

**Definition 3.1.1** Let \((E, \mathcal{E})\) be a measurable space. A transition kernel on \(E\) is a map \(P : E \times \mathcal{E} \to [0, 1]\) such that

i) for every \(x \in E\), the map \(B \mapsto P(x, B)\) is a probability measure on \((E, \mathcal{E})\),

ii) for every \(B \in \mathcal{E}\), the map \(x \mapsto P(x, B)\) is \(\mathcal{E}/\mathcal{B}\)-measurable.

Let \(X\) be an \((E, \mathcal{E})\) valued stochastic process defined on some underlying probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Then \(X\) is a Markov chain with initial distribution \(\nu\) if (i) \(P_\nu\{X_0 \in B\} = \nu(B)\) for all \(B \in \mathcal{E}\); (ii) if there exists a transition kernel \(P\) such that the Markov property holds:

\[
P_{\nu}\{X_{n+1} \in B \mid \mathcal{F}_n^X\} = P_{\nu}\{X_{n+1} \in B \mid \sigma(X_n)\} = P(X_n, B), \quad B \in \mathcal{E}, n = 0, 1, 2, \ldots \tag{3.1.3}
\]

**Remark** If \(E\) is a discrete space, and \(\mathcal{E}\) is the \(\sigma\)-algebra generated by the one-point sets, then for each set \(\{y\}, y \in E\) we write \(P(x, y)\) instead of \(P(x, \{y\})\). Moreover, \(P(x, B) = \sum_{y \in B} P(x, y)\), and so the transition kernel is completely specified by \(P(x, y), x, y \in E\).

As in the above, we would like to infer that

\[
P_{\nu}\{X_{n+m} \in B \mid \mathcal{F}_n^X\} = P^m(X_n, B), \quad B \in \mathcal{E}, n = 0, 1, 2, \ldots \tag{3.1.4}
\]
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where $P^m$ is defined inductively by

$$P^m(x, B) = \int P(y, B) P^{m-1}(x, dy), m = 2, 3, \ldots$$

For $m = 2$ we get (cf. (3.1.2))

$$P\{X_{n+2} \in B \mid \mathcal{F}^X_n\} = E_\nu(E_\nu(1_{\{B\}}(X_{n+2}) \mid \mathcal{F}^X_{n+1}) \mid \mathcal{F}^X_n) = E_\nu(P(X_{n+1}, B) \mid \mathcal{F}^X_n).$$

Our definition of the Markov property does not allow to infer (3.1.4) for $m = 2$ directly, which we do expect to hold. However, we can prove that it does. In fact, more general relations hold. Let us introduce some notation.

**Notation** Integrals of the form $\int fd\nu$ are often written in operator notation as $\nu f$. A similar notation for transition kernels is as follows. If $P(x, dy)$ is a transition kernel on a measurable space $(E, \mathcal{E})$ and $f$ is a non-negative (or bounded), measurable function on $E$, we define the function $Pf$ by

$$Pf(x) = \int f(y) P(x, dy). \quad (3.1.5)$$

Then $P(x, B) = \int 1_{\{B\}} P(x, dy) = P1_{\{B\}}(x)$. For notational convenience, write $b\mathcal{E}$ for the space of bounded, measurable functions $f : E \to \mathbb{R}$ and $m\mathcal{E}$ for the measurable functions $f : E \to \mathbb{R}$.

**Note** That $P f$ is bounded, for $f \in b\mathcal{E}$. Since $P$ is a transition kernel, $P1_{\{B\}} \in b\mathcal{E}$. Applying the standard machinery yields that $P f \in b\mathcal{E}$ for all $f \in b\mathcal{E}$, in particular $P f$ is $\mathcal{E}/\mathcal{E}$-measurable. In other words $P$ is a linear operator mapping $b\mathcal{E}$ to $b\mathcal{E}$.

Look up in BN section 3 Measurability what we mean by the ‘standard machinery’.

The Markov property (3.1.3) can now be reformulated as

$$E(1_{\{B\}}(X_{n+1}) \mid \mathcal{F}^X_n) = P1_{\{B\}}(X_n), \quad B \in \mathcal{E}, n = 0, 1, \ldots$$

Applying the standard machinery once more, yields

$$E(f(X_{n+1}) \mid \mathcal{F}^X_n) = Pf(X_n), \quad f \in b\mathcal{E}, n = 0, 1, \ldots$$

This has two consequences. The first is the immediate implication

$$P\{X_{n+2} \in B \mid \mathcal{F}^X_n\} = E(E(1_{\{B\}}(X_{n+2}) \mid \mathcal{F}^X_{n+1}) \mid \mathcal{F}^X_n)$$

$$= E(P1_{\{B\}}(X_{n+1}) \mid \mathcal{F}^X_n) = P(P1_{\{B\}})(X_n).$$

If $X_n = x$, the latter equals

$$\int_E P1_{\{B\}}(y) P(x, dy) = \int_E P(y, B) P(x, dy) = P^2(x, B).$$

In other words,

$$P\{X_{n+2} \in B \mid \mathcal{F}^X_n\} = P^2(X_n, B).$$

The second consequence is, that it makes sense to define the Markov property straightaway for bounded, measurable functions!
Continuous time Markov processes on a general state space  Let us now go to the continuous time case. Then we cannot define one stochastic matrix determining the whole probabilistic evolution of the stochastic process considered. Instead, we have a collection of transition kernels \((P_t)_{t \in T}\) that should be related through the so-called Chapman-Kolmogorov equation to allow the Markov property to hold.

**Definition 3.1.2** Let \((E, \mathcal{E})\) be a measurable space. A collection of transition kernels \((P_t)_{t \geq 0}\) is called a (homogeneous) transition function if for all \(s, t \geq 0, x \in E\) and \(B \in \mathcal{E}\)

\[
P_{t+s}(x, B) = \int P_s(x, dy) P_t(y, B).
\]

This relation is known as the **Chapman-Kolmogorov relation**.

Translated to operator notation, the Chapman-Kolmogorov equation states that for a transition function \((P_t)_{t \geq 0}\) it holds that for every non-negative (or bounded) measurable function \(f\) and \(s, t \geq 0\) we have

\[
P_{t+s}f = P_t(P_s f) = P_s(P_t f),
\]

cf. (3.1.5).

In other words, the linear operators \((P_t)_{t \geq 0}\) form a semigroup of operators on the space of non-negative (or bounded) functions on \(E\). In the sequel we will not distinguish between this semigroup and the corresponding (homogeneous) transition function on \((E, \mathcal{E})\), since there is a one-to-one relation between the two concepts.

**Further notation** Some further notation is enlightening. Let \(f, g, h\) be bounded (non-negative) measurable functions on \(E\). As argued before, \(P_t f\) is bounded, \(\mathcal{E}/\mathcal{E}\)-measurable. Hence multiplying by \(g\) gives \(gP_t f\), which is bounded, measurable. Here

\[
gP_t f(x) = g(x) \cdot P_t f(x) = g(x) \int f(y) P_t(x, dy).
\]

Then we can apply \(P_s\) to this function, yielding the bounded, measurable function \(P_s g P_t f\), with

\[
P_s g P_t f(x) = \int g(y) P_t f(y) P_s(x, dy) = \int g(y) \int f(z) P_t(y, dz) P_s(x, dy).
\]

\(hP_s g P_t f\) is again bounded, measurable and we can integrate over the probability distribution \(\nu\) on \((E, \mathcal{E})\):

\[
\nu h P_s g P_t f = \int h(x) P_s g P_t f(x) \nu(dx) = \int h(x) \int g(y) \int f(z) P_t(y, dz) P_s(x, dy) \nu(dx).
\]

To summarise, we get the following alternative notation for Markov processes that will be interchangedly used.
We can now give the definition of a Markov process.

\[ P_x \{ X_s \in A \} = \mathbb{E}_x 1_A(X_s) = \mathbb{E}_x \mathbb{E}_x(1_A(X_{s+0}) \mid \mathcal{F}_0) = \mathbb{E}_x P_s 1_A(X_0) = P_s(1_A(x)) = \int_{y \in E} 1_A(y) P_s(x, dy) = P_s(x, A). \quad (3.1.6) \]

This can be generalised to: for \( f \in b\mathcal{E} \)

\[ \mathbb{E}_x f(X_t) = \int f(y) P_t(x, dy) = P_t f(x), \]

and

\[ \mathbb{E}_\nu f(X_t) = \int_x \int_y f(y) P_t(x, dy) \nu(dx) = \int_x P_t f(x) \nu(dx) = \nu P_t f. \]

Note that by the ‘standard machinery’, we may replace bounded \( f \) by non-negative \( f \) in the definition.

We can also let the initial distribution be given by the value \( X_s \) for some \( s \geq 0 \). This results in the following notation. For \( f \in b\mathcal{E} \)

\[ \mathbb{E}_{X_s} f(X_t) = \int f(y) P_t(X_s, dy) = P_t f(X_s), \]

which we understand as being equal to \( \mathbb{E}_x f(X_t) = P_t f(x) \) on the event \( \{ X_s = x \} \).

We can now give the definition of a Markov process.

**Definition 3.1.3** Let \((E, \mathcal{E})\) be a measurable space and let \( X \) be an \((E, \mathcal{E})\)-valued stochastic process that is adapted to some underlying filtered space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P}_\nu)\). \( X \) is a **Markov process with initial distribution** \( \nu \), if

i) \( \mathbb{P}_\nu \{ X_0 \in B \} = \nu(B) \) for every \( B \in \mathcal{E} \);

ii) **(Markov property)** there exists a transition function \((P_t)_t\), such that for all \( s, t \geq 0 \) and \( f \in b\mathcal{E} \)

\[ \mathbb{E}_\nu(f(X_{t+s}) \mid \mathcal{F}_s) = P_t f(X_s) = \mathbb{E}_{X_s} f(X_t) \]

\( \mathbb{P}_\nu \)-a.s. \quad (3.1.7) \]

**Definition 3.1.4** Let \((E, \mathcal{E})\) be a measurable space and let \( X : (\Omega, \mathcal{F}) \to (E^R, \mathcal{E}^R) \) be a map that is adapted to the filtration \((\mathcal{F}_t)_t\), with \( \mathcal{F}_t \subset \mathcal{F}, t \geq 0 \). \( X \) is a **Markov process**, if there exists a transition function \((P_t)_t\), such that for each distribution \( \nu \) on \((E, \mathcal{E})\) there exists a probability distribution \( \mathbb{P}_\nu \) on \((\Omega, \mathcal{F})\) with

i) \( \mathbb{P}_\nu \{ X_0 \in B \} = \nu(B) \) for every \( B \in \mathcal{E} \);

ii) **(Markov property)** for all \( s, t \geq 0 \) and \( f \in b\mathcal{E} \)

\[ \mathbb{E}_\nu(f(X_{t+s}) \mid \mathcal{F}_s) = P_t f(X_s) = \mathbb{E}_{X_s} f(X_t), \quad \mathbb{P}_\nu \)-a.s. \quad (3.1.8) \]
A main question is whether such processes exist, and whether sufficiently regular versions of these processes exist. As in the first chapter we will address this question by first showing that the fdd’s of a Markov process (provided it exists) are determined by transition function and initial distribution. You have to realise further that a stochastic process with a transition function \((P_t)\) need not be Markov in general. The Markov property really is a property of the underlying stochastic process (cf. Example 3.1.1).

**Lemma 3.1.5** Let \(X\) be an \((E, \mathcal{E})\)-valued stochastic process with transition function \((P_t)_{t \geq 0}\), adapted to the filtration \((\mathcal{F}_t)_t\). Let \(\nu\) be a distribution on \((E, \mathcal{E})\).

If \(X\) is Markov with initial distribution \(\nu\), then for all \(0 = t_0 < t_1 < \cdots < t_n\), and all functions \(f_0, \ldots, f_n \in b\mathcal{E}, \ n \in \mathbb{Z}_+\),

\[
E_\nu \prod_{i=0}^{n} f_i(X_{t_i}) = \nu f_0 P_{t_1-t_0} f_1 \cdots P_{t_n-t_{n-1}} f_n.
\]

(3.1.9)

Vice versa, suppose that (3.1.9) holds for all \(0 = t_0 < t_1 < \cdots < t_n\), and all functions \(f_0, \ldots, f_n \in b\mathcal{E}, n \in \mathbb{Z}_+\). Then \(X\) is Markov with respect to the natural filtration \((\mathcal{F}_t^X)_t\).

In either case, (3.1.9) also holds for non-negative functions \(f_0, \ldots, f_n \in \mathcal{E}\).

**Remark:** the proof of the Lemma shows that it is sufficient to check (3.1.9) for indicator functions. For indicator functions (3.1.9) is nothing but the fdd corresponding to time points \(t_0, \ldots, t_n\) on a \(\pi\)-system.

**Proof.** Let \(X\) be a Markov process with initial distribution \(\nu\). Then

\[
E_\nu \prod_{i=0}^{n} f_i(X_{t_i}) = E_\nu \left( \prod_{i=0}^{n} f_i(X_{t_i}) \mid \mathcal{F}_{t_{n-1}} \right)
\]

\[
= E_\nu \prod_{i=0}^{n-1} f_i(X_{t_i}) E_\nu (f(X_{t_n}) \mid \mathcal{F}_{t_{n-1}})
\]

\[
= E_\nu \prod_{i=0}^{n-1} f_i(X_{t_i}) P_{t_n-t_{n-1}} f_n(X_{t_{n-1}}).
\]

Now, \(P_{t_n-t_{n-1}} f_n \in b\mathcal{E}\), and so one has

\[
E_\nu \prod_{i=0}^{n-1} f_i(X_{t_i}) P_{t_n-t_{n-1}} f_n(X_{t_{n-1}}) = E_\nu \prod_{i=0}^{n-2} f_i(X_{t_i}) E_\nu (f_{n-1}(X_{t_{n-1}}) P_{t_n-t_{n-1}} f_n(X_{t_{n-1}}) \mid \mathcal{F}_{t_{n-2}})
\]

\[
= E_\nu \prod_{i=0}^{n-2} f_i(X_{t_i}) P_{t_{n-1}-t_{n-2}} f_{n-1} P_{t_n-t_{n-1}} f_n(X_{t_{n-2}}).
\]

Iterating this yields

\[
E_\nu \prod_{i=0}^{n} f_i(X_{t_i}) = E_\nu f_0(X_{t_0}) P_{t_1-t_0} f_1 P_{t_2-t_1} f_1 \cdots P_{t_n-t_{n-1}} f_n(X_0)
\]

\[
= \nu f_0 P_{t_1-t_0} f_1 P_{t_2-t_1} f_1 \cdots P_{t_n-t_{n-1}} f_n.
\]
Conversely, assume that (3.1.9) holds for all 0 = t_0 < t_1 < \cdots < t_n, all functions f_0, \ldots, f_n \in b\mathcal{E}. We have to show that (i) P_\nu \{ X_0 \in B \} = \nu (B) for all B \in \mathcal{E}, and that (ii) for any s, t \geq 0, all sets A \in \mathcal{F}_s^X$

$$E_\nu 1_{\{ A \}} f(X_{t+s}) = E_\nu 1_{\{ A \}} P_t f(X_s).$$ (3.1.10)

Let B \in \mathcal{E}, put n = 0, f_0 = 1_{\{ B \}}. (i) immediately follows.

We will show (ii). To derive (3.1.10), it is sufficient to check this for a \pi-system containing \Omega and generating \mathcal{F}_s^X. As the \pi-system we take

$$\left\{ A = \{ X_{t_0} \in A_0, X_{t_1} \in A_1, \ldots X_{t_n} \in A_n \} \mid t_0 = 0 < t_1 < \cdots < t_n \leq s, \right.$$ 

$$A_i \in \mathcal{E}, i = 0, \ldots, n, n = 0, \ldots \right\}$$

Let $f_i = 1_{\{ A_i \}}$, then $\prod_{i=0}^n f_i(X_{t_i}) = 1_{\{ \lambda_{t_0} \in A_0, \lambda_{t_1} \in A_1, \ldots, \lambda_{t_n} \in A_n \}}$ and so, assuming that $t_n < s$

$$E_\nu \prod_{i=0}^n 1_{\{ A_i \}}(X_{t_i}) 1_{\{ E \}}(X_s) f(X_{t+s}) = \nu 1_{\{ A_0 \}} P_{t_1-t_0} 1_{\{ A_1 \}} P_{t_2-t_1} \cdots P_{t_n-t_{n-1}} 1_{\{ A_n \}} P_{t+s-t_n} f

= \nu 1_{\{ A_0 \}} P_{t_1-t_0} 1_{\{ A_1 \}} \cdots P_{e-t_n} (P_t f)$

$$= E_\nu \prod_{i=0}^n 1_{\{ A_i \}}(X_{t_i}) (P_t f)(X_s),$$

which we wanted to prove. The reasoning is similar if $t_n = s$.

This implies that (3.1.10) holds for all sets A in a \pi-system generating \mathcal{F}_s^X, hence it holds for \mathcal{F}_s^X. Consequently, $E_\nu (f(X_{t+s}) \mid \mathcal{F}_s^X) = P_t f(X_s)$, a.s. QED

**Example 3.1.1 (Not a Markov process)** Consider the following space

$$\mathcal{S} = \{ (1, 1, 1), (2, 2, 2), (3, 3, 3), (1, 2, 3), (1, 3, 2), (2, 3, 1), (2, 1, 3), (3, 1, 2), (3, 2, 1) \},$$

with \sigma-algebra $\mathcal{S} = 2^\mathcal{S}$. Putting $P\{ x \} = 1/9$ defines a probability measure on this space.

Define a sequence of i.i.d. random vectors $Z_k = (X_{3k}, X_{3k+1}, X_{3k+2})$, $k = 0, \ldots$ on $(\mathcal{S}, \mathcal{S}, P)$. Then the sequence $\{ X_n \}_{n}$ is an $(E = \{ 1, 2, 3 \}, \mathcal{E} = 2^E)$-valued stochastic process on $(\mathcal{S}, \mathcal{S}, P)$ in discrete time. Then $P\{ X_{n+1} = j \mid \sigma(X_n) \} = 1/3$ for each $j \in \{ 1, 2, 3 \}$ and $n$, meaning that the motion is determined by the $3 \times 3$ stochastic matrix with all elements equal to $1/3$.

Let $\{ \mathcal{F}_n^X = \sigma(X_k, k \leq n) \}_{n}$ be the natural filtration generated by $\{ X_n \}_{n}$. $\{ X_n \}_{n}$ is not a Markov chain w.r.t $\{ \mathcal{F}_n^X \}_{n}$, since $P\{ X_2 = 1 \mid \sigma(X_0, X_1) \} = f(X_0, X_1)$ with

$$f(X_0, X_1) = \begin{cases} 1, & (X_0, X_1) \in \{ (1, 1), (2, 3), (3, 2) \} \\ 0, & \text{otherwise}. \end{cases}$$

Hence $f(1, 1) \neq f(2, 1)$, thus showing that the Markov property lacks.

**Example 3.1.2 (A (BM process))** Let $W$ be a standard BM on an underlying probability space $(\Omega, \mathcal{F}, P)$. Let $X_0$ be a measurable random variable with distribution $\nu = \delta_x$, for some $x \in \mathbb{R}$, independent of $W$. Define $X_t = X_0 + W_t$, $t \geq 0$. Then $X = (X_t)_t$ is a Markov process
with initial distribution \( \nu \) with respect to its natural filtration. Note that \( X_t - X_s = W_t - W_s \) is independent of \( \mathcal{F}_s^X \).

To see that \( X \) is a Markov process, let \( f \) be a bounded, measurable function (on \( \mathbb{R} \)). Write \( Y_t = W_{t+s} - W_s \). Then \( Y_t \overset{d}{=} N(0, t) \) is independent of \( \mathcal{F}_s^X \) and so with BN Lemma 7.5

\[
E_\nu(f(X_{t+s}) | \mathcal{F}_s^X) = E_\nu(f(Y_t + W_s + x) | \mathcal{F}_s^X) = g(x)
\]

for the function \( g \) given by

\[
g(z) = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi t}} f(y + z)e^{-y^2/2t} dy
= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi t}} f(y)e^{-(y-z)^2/2t} dy
= P_t f(z)
\]

with \( P_t \) defined by

\[
P_t f(z) = \int f(y)p(t, z, y) dy,
\]

where

\[
p(t, z, y) = \frac{1}{\sqrt{2\pi t}} e^{-(y-z)^2/2t}.
\]

Hence

\[
E(f(X_{t+s}) | \mathcal{F}_s^X) = g(X_s) = P_t f(X_s) \quad \text{a.s.}
\]

It is easily shown that \( P_t \) is a transition function. Fix \( t \geq 0 \). Measurability of \( P_t(x, B) \) in \( x \) for each set \( B = (-\infty, b), b \in \mathbb{R} \), follows from continuity arguments. Together with \( \mathbb{R} \), these sets form a \( \pi \)-system for \( \mathcal{B} \). Then apply the \( d \)-system recipe to the set \( \mathcal{S} = \{B \in \mathcal{B} | x \mapsto P_t(x, B) \text{is } \mathcal{E}/B[0,1]-\text{measurable}\} \).

**Example 3.1.3 ((B) Ornstein-Uhlenbeck process)** Let \( W \) be a standard Brownian motion. Let \( \alpha, \sigma^2 > 0 \) and let \( X_0 \) be a \( \mathbb{R} \)-valued random variable with distribution \( \nu \) that is independent of \( \sigma(W_t, t \geq 0) \). Define the scaled Brownian motion by

\[
X_t = e^{-\alpha t}(X_0 + W_{\sigma^2(\exp(2\alpha t) - 1)}/2\alpha).
\]

If \( \nu = \delta_x, X = (X_t)_t \) a Markov process with the \( P_\nu \) distribution of \( X_t \) a normal distribution with mean \( \exp\{-\alpha t\}x \) and variance \( \sigma^2(1 - e^{-2\alpha t})/2\alpha \). Note that \( X_t \overset{D}{\rightarrow} N(0, \sigma^2/2\alpha) \).

If \( X_0 \overset{d}{=} N(0, \sigma^2/2\alpha) \) then \( X_t \) is a Gaussian, Markov process with mean \( m(t) = 0 \) and covariance function \( r(s,t) = \sigma^2 \exp\{-\alpha|t-s|\}/2\alpha \). See Exercise 3.1.

**Example 3.1.4 ((C) Geometric Brownian motion)** Let \( W \) be a standard BM on an underlying probability space \((\Omega, \mathcal{F}, P)\). Let \( X_0 \) be a measurable positive random variable with distribution \( \nu \), independent of \( W \). Let \( \mu \in \mathbb{R} \) and \( \sigma^2 \in (0, \infty) \) and define \( X_t = X_0 e^{(\mu - \sigma^2/2)t + \sigma W_t}, t \geq 0 \). Then \( X \) is a Markov process. Depending on the value \( \mu \) it is a (super/sub) martingale. Geometric Brownian motion is used in financial mathematics to model stock prices.
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Example 3.1.5 (Poisson process) Let $N$ be a Poisson process on an underlying probability space $(\Omega, \mathcal{F}, P)$. Let $X_0$ be a measurable random variable with distribution $\nu = \delta_x$, for some $x \in \mathbb{Z}_+$, independent of $N$. Define $X_t = X_0 + N_t$, $t \geq 0$. Then $X = (X_t)_{t \geq 0}$ is a Markov process with initial distribution $\nu$, w.r.t. the natural filtration.

This can be shown in precisely the same manner as for BM (example 3.1.2A). In this case the transition function $P_t$ is a stochastic matrix, $t \geq 0$, with

$$P_t(x, y) = P\{N_t = y - x\}, \quad y \geq x.$$  

(cf. Exercise 3.8).

In general it is not true that a function of a Markov process with state space $(E, \mathcal{E})$ is a Markov process. The following lemma gives a sufficient condition under which this is the case.

Lemma 3.1.4 Let $X$ be a Markov process with state space $(E, \mathcal{E})$, initial distribution $\nu$ and transition function $(P_t)_t$, defined on an underlying filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$. Suppose that $(E', \mathcal{E}')$ is a measurable space and let $\phi : E \to E'$ be measurable and onto. If $(Q_t)_t$ is a collection of transition kernels such that

$$P_t(f \circ \phi) = (Q_t f) \circ \phi$$

for all bounded, measurable functions $f$ on $E'$, then $Y = \phi(X)$ is a Markov process with respect to $(\mathcal{F}_t)_{t \geq 0}$, with state space $(E', \mathcal{E}')$, initial measure $\nu'$, with $\nu'(B') = \nu(\phi^{-1}(B'))$, $B' \in \mathcal{E}'$, and transition function $(Q_t)_t$.

Proof. Let $f$ be a bounded, measurable function on $E'$. By assumption and the semi-group property of $(P_t)_t$,

$$(Q_t Q_s f) \circ \phi = P_t((Q_s f) \circ \phi) = P_t P_s (f \circ \phi) = P_{t+s} (f \circ \phi) = (Q_{t+s} f) \circ \phi.$$  

Since $\phi$ is onto, this implies that $(Q_t)_t$ is a semigroup. It is easily verified that $Y$ has the Markov property (see Exercise 3.3). QED

Example 3.1.6 ($W_t^2$ is a Markov process) We apply Lemma 3.1.6. In our example one has the function $\phi : E = \mathbb{R} \to E' = \mathbb{R}_+$ given by $\phi(x) = x^2$. The corresponding $\sigma$-algebras are simply the Borel-$\sigma$-algebras on the respective spaces.

If we can find a transition kernel $Q_t$, $t \geq 0$, such that

$$P_t(f \circ \phi)(x) = (Q_t f) \circ \phi(x), x \in \mathbb{R}$$  

(3.1.11)

for all bounded, measurable functions $f$ on $E' = \mathbb{R}_+$, then $\phi(W_t) = W_t^2$, $t \geq 0$, is a Markov process (w.r.t. its natural filtration).

Let $f$ be a bounded, measurable function on $\mathbb{R}_+$. Then for $x \in \mathbb{R}$

$$P_t(f \circ \phi)(x) = \int_{-\infty}^{\infty} p(t, x, y) f(y^2) dy$$

$$= \int_0^{\infty} (p(t, x, y) + p(t, x, -y)) f(y^2) dy$$

$$u = y^2 \Rightarrow y = \sqrt{u}, dy = du/2\sqrt{u}$$

$$= \int_0^{\infty} p(t, x, \sqrt{u} + p(t, x, -\sqrt{u}) \frac{1}{2\sqrt{u}} f(u) du.$$  

(3.1.12)
Define for $y \in \mathbb{R}_+$, $B \in \mathcal{E}' = \mathcal{B}(\mathbb{R}_+)$

$$Q_t(y, B) = \int_B (p(t, \sqrt{y}, \sqrt{u}) + p(t, \sqrt{y}, -\sqrt{u})) \frac{1}{2\sqrt{u}} du.$$ 

One can check that $(Q_t)_{t \geq 0}$, is a transition kernel. Moreover, from (3.1.12) it follows for $x \in \mathbb{R}_+$ that

$$(Q_t f) \circ \phi (x) = (Q_t f)(x^2) = P_t(f \circ \phi)(x).$$

For $x < 0$ one has $p(t, x, y) + p(t, x, -y) = p(t, -x, y) + p(t, -x, -y)$ and so $P_t(f \circ \phi)(x) = P_t(f \circ \phi)(-x)$. Since $(Q_t f) \circ \phi (x) = (Q_t f)(x^2) = (Q_t f) \circ \phi (-x)$, the validity of (3.1.11) follows immediately.

### 3.2 Existence of a canonical version

The question is whether we can construct processes satisfying definition 3.1.3. In this section we show that this is indeed the case. In other words, for a given transition function $(P_t)_t$ and probability measure $\nu$ on a measurable space $(E, \mathcal{E})$, we can construct a so-called canonical Markov process $X$ which has initial distribution $\nu$ and transition function $(P_t)_t$. We go back to the construction in Chapter 1. Note that, although we consider processes in continuous time, the results are valid as well for discrete time processes.

Recall that an $E$-valued process can be viewed as a random variable of the space $E^{\mathbb{R}^+_0}$ of $E$-valued functions $f$ on $\mathbb{R}_+$, or of a subspace $\Gamma \subset E^{\mathbb{R}^+_0}$ if $X$ is known to have more structure. The $\sigma$-algebra $\Gamma \cap \mathcal{E}^{\mathbb{R}^+_0}$ is the smallest $\sigma$-algebra that makes all projections $f \to f(t)$ measurable.

As in Chapter 1, let $\Omega = \Gamma$ and $\mathcal{F} = \Gamma \cap \mathcal{E}^{\mathbb{R}^+_0}$. Consider the process $X = (X_t)_{t \geq 0}$ defined as the identity map

$$X(\omega) = \omega,$$

so that $X_t(\omega) = \omega_t$ is projection on the $t$-th coordinate. By construction $X : (\Omega, \mathcal{F}) \to (\Omega, \mathcal{F})$ and $X_t : (\Omega, \mathcal{F}) \to (E, \mathcal{E})$ are measurable maps. The latter implies that $X$ is a stochastic process in the sense of Definition 1.1.1. $X$ is adapted to the natural filtration $(\mathcal{F}_t^X = \Gamma \cap \mathcal{E}^{[0,t]}_t)$. In a practical context, the path space, or a subspace, is the natural space to consider as it represents the process itself evolving in time.

Note that we have not yet defined a probability measure on $(\Omega, \mathcal{F})$. The Kolmogorov consistency theorem 1.2.3 validates the existence of a process on $(\Omega, \mathcal{F})$ with given fdds. Hence, we have to specify appropriate fdds based on the given transition function $(P_t)_t$ and initial distribution $\nu$.

In order to apply this theorem, from this point on we will assume that $(E, \mathcal{E})$ is a Polish space, endowed with its Borel $\sigma$-algebra.

**Corollary 3.2.2 (to the Kolmogorov consistency theorem)** Let $(P_t)_t$ be a transition function and let $\nu$ be a probability measure on $(E, \mathcal{E})$. Then there exists a unique probability measure $P_\nu$ on $(\Omega, \mathcal{F})$ such that under $P_\nu$ the canonical process $X$ is a Markov process with initial distribution $\nu$ and transition function $(P_t)_t$ with respect to its natural filtration $(\mathcal{F}_t^X)_t$. 

3.2. EXISTENCE OF A CANONICAL VERSION

Proof. For any \( n \) and all \( 0 = t_0 < t_1 < \cdots < t_n \) we define a probability measure on \((E^{n+1},\mathcal{E}^{n+1})\) (see Exercise 3.6) by

\[
\mu_{t_0,\ldots,t_n}(A_0 \times A_1 \times \cdots \times A_n) = \nu 1_{\{A_0\}} P_{t_1-t_0} 1_{\{A_1\}} \cdots P_{t_n-t_{n-1}} 1_{\{A_n\}}, \quad A_0,\ldots,A_n \in \mathcal{E},
\]

and on \((E^n,\mathcal{E}^n)\) by

\[
\mu_{t_1,\ldots,t_n}(A_1 \times \cdots \times A_n) = \nu 1_{\{E\}} P_{t_1-t_0} 1_{\{A_1\}} \cdots P_{t_n-t_{n-1}} 1_{\{A_n\}}, \quad A_1,\ldots,A_n \in \mathcal{E}.
\]

By the Chapman-Kolmogorov equation these probability measures form a consistent system (see Exercise 3.6). Hence by Kolmogorov’s consistency theorem there exists a probability measure \( P_{\nu} \) on \((\Omega,\mathcal{F})\), such that under \( P_{\nu} \) the measures \( \mu_{t_1,\ldots,t_n} \) are precisely the fdd’s of the canonical process \( X \).

In particular, for any \( n, 0 = t_0 < t_1 < \cdots < t_n \), and \( A_0,\ldots,A_n \in \mathcal{E} \)

\[
P\{X_{t_0} \in A_0,\ldots,X_{t_n} \in A_n\} = \nu 1_{\{A_0\}} P_{t_1-t_0} \cdots P_{t_n-t_{n-1}} 1_{\{A_n\}}.
\]

By virtue of the remark following Lemma 3.1.5 this implies that \( X \) is Markov w.r.t. its natural filtration. \( \text{QED} \)

As the initial measure \( \nu \) we can choose the Dirac measure \( \delta_x \) at \( x \in E \). By the above there exists a measure \( P_{\nu} \) on \((\Omega,\mathcal{F})\), such that the canonical process \( X \) has distribution \( P_{\nu} \). This distributions has all mass on paths \( \omega \) starting at \( x: \omega_0 = x \). In words, we say that under \( P_{\nu} \) the process \( X \) starts at point \( x \). Note that (cf. (3.1.6))

\[
P_{\nu}\{X_t \in A\} = P_t(x, A) = \int P_t(y, A) \delta_x(dy)
\]

is a measurable function in \( x \). In particular, since any distribution \( \nu \) can be obtained as a convex combination of Dirac measures, we get

\[
P_{\nu}\{X_t \in A\} = \int P_t(y, A) \nu(dy) = \int P_y\{X_t \in A\} \nu(dy).
\]

Similarly, the fdd’s of \( X \) under \( P_{\nu} \) can be written as convex combination of the fdd’s of \( X \) under \( P_x \), \( x \in E \). The next lemma shows that this applies to certain functions of \( X \) as well.

Lemma 3.2.3 Let \( Z \) be an \( \mathcal{F}_X^\infty \)-measurable random variable, that is either non-negative or bounded. Then the map \( x \to \mathbb{E}_x Z \) is \( \mathcal{E}/\mathcal{B} \)-measurable and for every initial distribution \( \nu \)

\[
\mathbb{E}_{\nu} Z = \int_x \mathbb{E}_x Z \nu(dx).
\]

Review BN §3 on monotone class theorems

Proof. Consider the collection of sets

\[
\mathcal{S} = \{ \Gamma \in \mathcal{F}_X^\infty \mid x \to \mathbb{E}_x 1_{\{\Gamma\}} \text{ is measurable and } \mathbb{E}_{\nu} 1_{\{\Gamma\}} = \int \mathbb{E}_x 1_{\{\Gamma\}} \nu(dx) \}.
\]
It is easily checked that this is a \( d \)-system. The collection of sets
\[
\mathcal{G} = \{ \{ X_{t_1} \in A_1, \ldots, X_{t_n} \in A_n \} \mid A_1, \ldots, A_n \in \mathcal{E}, 0 \leq t_1 < \cdots < t_n, n \in \mathbb{Z}_+ \}
\]
is a \( \pi \)-system for \( \mathcal{F}_\infty^X = \mathcal{E}^{\mathbb{R}_+} \). So if we can show that \( \mathcal{G} \subseteq \mathcal{S} \), then by BN Lemma 3.8 \( \mathcal{F}_\infty^X \subseteq \mathcal{S} \). But this follows from Lemma 3.1.5.

It follows that the statement of the lemma is true for \( Z = 1_{\{ \Gamma \}}, \Gamma \in \mathcal{F}_\infty^X \). Apply the standard machinery to obtain the validity of the lemma for \( \mathcal{F}_\infty^X \)-measurable bounded or non-negative random variables \( Z \). See also Exercise 3.7.

This lemma allows to formulate a more general version of the Markov property. For any \( t \geq 0 \) we define the translation or shift operator \( \theta_t : \mathbb{E}^{\mathbb{R}_+} \to \mathbb{E}^{\mathbb{R}_+} \) by
\[
(\theta_t(\omega))_s = \omega_{t+s}, \quad s \geq 0, \quad \omega \in \mathbb{E}^{\mathbb{R}_+}.
\]
So \( \theta_t \) just cuts off the part of \( \omega \) before time \( t \) and shifts the remainder to the origin. Clearly \( \theta_t \circ \theta_s = \theta_{t+s} \).

---

**Theorem 3.2.4 (Generalised Markov property for canonical process)** Assume that \( X \) is a canonical Markov process on \( (\Omega = \mathbb{E}^{\mathbb{R}_+}, \mathcal{F} = \mathcal{E}^{\mathbb{R}_+} \cap \Gamma) \). In other words, for each distribution \( \nu \) on \( (\mathbb{E}, \mathcal{E}) \), there exists a probability distribution \( \nu^\mathbb{E} \) on \( (\Omega, \mathcal{F}) \), such that \( X \) is the canonical Markov process on \( (\Omega, \mathcal{F}, \nu^\mathbb{E}) \) with initial distribution \( \nu \).

Note that \( \mathcal{F}_t^X = \mathcal{E}^{[0,t]} \cap \Gamma \) and \( \theta_t \) is \( \mathcal{F} \)-measurable for every \( t \geq 0 \) (why?).

---

**Proof.** Fix an initial probability measure \( \nu \). We will first show that (3.2.1) holds for all \( Z = 1_{\{B\}}, B \in \mathcal{F}_\infty^X \). Let
\[
\mathcal{S} = \{ B \in \mathcal{F}_\infty^X \mid \mathbb{E}_\nu(1_{\{B\}} \circ \theta_t | \mathcal{F}_t) = \mathbb{E}_X \theta_t 1_{\{B\}}, \quad \nu^\mathbb{E} \text{ - a.s.} \}.
\]
Then \( \mathcal{S} \) is a \( \pi \)-system, since (i) \( \Omega \in \mathcal{S} \), (ii) \( B, B' \in \mathcal{S} \), \( B \subseteq B' \), implies \( B' \setminus B \in \mathcal{S} \), and (iii) for \( B_0, n = 1, \ldots, \) in \( \mathcal{F}_\infty^X \) a non-decreasing sequence of sets with \( B_n \in \mathcal{S} \), \( n = 1, 2, \ldots \), one has union of \( B_n \) is also in \( \mathcal{S} \). Indeed, (ii) and (iii) follow from linearity of integrals and monotone convergence.

Recall that collection of all finite-dimensional rectangles, \( \mathcal{A} \) say, is a \( \pi \)-system generating \( \mathcal{F}_\infty^X \). Note that \( B \in \mathcal{A} \) whenever there exist \( n \in \mathbb{Z}_+ \), \( 0 = s_0 < s_1 < \cdots < s_m \), \( B_0, \ldots, B_m \in \mathcal{E} \), \( n \in \mathbb{Z}_+ \), such that \( B = \{ X_{s_0} \in B_0, \ldots, X_{s_m} \in A_m \} \). If we can show that \( \mathcal{A} \subseteq \mathcal{S} \), then by BN Lemma 3.4 it follows that \( \sigma(\mathcal{A}) = \mathcal{F}_\infty^X \subseteq \mathcal{S} \).
3.3. STRONG MARKOV PROPERTY

Take a finite-dimensional rectangle $B = \{X_{s_1} \in B_1, \ldots, X_{s_m} \in B_m\}$, where $0 \leq s_1 < \cdots < s_m$, $B_i \in \mathcal{E}$, $i = 1, \ldots, m$. Using Lemma 3.1.5, it follows that

$$
\mathbb{E}_\nu(1_B \circ \theta_t | \mathcal{F}_t) = \mathbb{E}_\nu(\prod_{i=1}^{m} 1_{\{B_i\}}(X_{s_i}) \circ \theta_t | \mathcal{F}_t)
$$

$$
= \mathbb{E}_\nu(\prod_{i=1}^{m} 1_{\{B_i\}}(X_{t+s_i}) | \mathcal{F}_t)
$$

$$
= \mathbb{E}_\nu(1_{\{B_1\}} P_{s_2-s_1} 1_{\{B_2\}} \cdots P_{s_m-s_{m-1}} 1_{\{B_m\}} P_{s_m-s_{m-1}} 1_{\{B_m\}}(X_{t+s_1}) | \mathcal{F}_t)
$$

$$
= P_{s_1} 1_{\{B_2\}} P_{s_2-s_1} 1_{\{B_3\}} \cdots P_{s_m-s_{m-1}} 1_{\{B_m\}} P_{s_m-s_{m-1}} 1_{\{B_m\}}(X_t)
$$

$$
= \mathbb{E}_{X_t} \prod_{i=1}^{m} 1_{\{B_i\}}(X_{s_i})
$$

where we have consecutively conditioned on the $\sigma$-algebras $\mathcal{F}_{t+s_{m-1}}, \ldots, \mathcal{F}_{t+s_1}$, and used the Markov property. For the last equality we have used (3.1.9).

This results in having proved (3.2.1) for indicator functions $Z$. Apply the standard machinery to prove it for step functions (by linearity of integrals), non-negative functions, and bounded functions $Z$.

QED

We end this section with an example of a Markov process with a countable state space.

Example 3.2.1 (Markov jump process) Let $E$ be a countable state space with $\sigma$-algebra $\mathcal{E} = 2^E$ generated by the one-point sets. Let $P$ be an $E \times E$ stochastic matrix. We define the transition function $(P_t)_t$ as follows:

$$
P_t(x,y) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} P^{(n)}(x,y), \quad x,y \in E
$$

where $P^{(n)} = (P)^n$ is the $n$-th power of $P$, and $P^{(0)} = I$ is the identity matrix.

By virtue of Corollary 3.2.2 the canonical process $X$ on $(E^{R_+}, \mathcal{E}^{R_+})$ with initial distribution $\nu$ is a Markov process with respect to its natural filtration.

The construction is as follows. Construct independently of $X_0$, a Poisson process $N$ (cf. Chapter 1), starting at 0 and, independently, a discrete-time Markov chain $Y$ with transition matrix $P$, with initial distribution $\delta_x$. If $N_t = n$, then $X_t = Y_n$. Formally $X_t = \sum_{n=0}^{\infty} 1_{\{N_t=n\}} Y_n$. By construction $X_t$ has right-continuous paths.

3.3 Strong Markov property

3.3.1 Strong Markov property

Let $X$ be an $(E, \mathcal{E})$-valued Markov process on $(\Omega, \mathcal{F})$, adapted to the filtration $(\mathcal{F}_t)_t$ and transition function $(P_t)_t$. Assume that $X$ has everywhere right-continuous paths, and that $E$ is a Polish space, with $\mathcal{E}$ the Borel-$\sigma$-algebra.

Definition 3.3.1 $X$ is said to have the strong Markov property if for every function $f \in b\mathcal{E}$, any adapted stopping time $\sigma$ and any initial distribution $\nu$ and any $t \geq 0$

$$
1_{\{\sigma < \infty\}} \mathbb{E}_\nu(f(X_{\sigma+t}) | \mathcal{F}_\sigma) = 1_{\{\sigma < \infty\}} \mathbb{E}_{X_\sigma}(f(X_t)), \quad P_\nu \text{ a.s.}
$$

(3.3.1)
Note that we have to exclude the event \( \{ \sigma = \infty \} \), since \( X_t \) may not have a limit as \( t \to \infty \) and hence it may not be possible to define the value \( X_\infty \) appropriately.

**Lemma 3.3.2** Let \( X \) be an \((E, \mathcal{E})\)-valued Markov process. Then (3.3.1) holds for any function \( f \in b\mathcal{E} \), any initial distribution \( \nu \) and any stopping time \( \sigma \), for which there exists a countable subset \( S \subset [0, \infty) \) such that \( \sigma \in S \cup \{ \cdot \} \).

**Proof.** Any stopping time is optional, and so by the fact that \( \sigma \) is countably valued, it is easily checked that \( A \in \mathcal{F}_\sigma \) if and only if \( A \cap \{ \sigma = s \} \in \mathcal{F}_s \) for each \( s \in S \).

It is directly checked that \( 1_{\{ \sigma < \infty \}} X_\sigma \) is \( \mathcal{F}_\sigma \)-measurable. Use Lemma 3.2.3 to derive that the map \( \omega \mapsto 1_{\{ \sigma(\omega) < \infty \}} E_{X_\omega}(f(X_t)) \) is \( \mathcal{F}_\sigma / \mathcal{B} \)-measurable as a composition of measurable maps. The next step is to show that

\[
E_\nu 1_{\{ \sigma < \infty \}} f(X_{\sigma + t}) = E_\nu 1_{\{ \sigma < \infty \}} E_{X_\sigma} f(X_t), \quad A \in \mathcal{F}_\sigma.
\]

If \( A \in \mathcal{F}_\sigma \) with \( A \subset \{ \sigma = s \} \) for some \( s \in S \), then \( A \in \mathcal{F}_s \). By the Markov property Theorem 3.2.4

\[
E_\nu 1_{\{ \sigma < \infty \}} f(X_{\sigma + t}) = E_\nu 1_{\{ \sigma < \infty \}} f(X_{\sigma + t}) = E_\nu 1_{\{ \sigma < \infty \}} E_{X_\sigma} f(X_t) = E_\nu 1_{\{ \sigma < \infty \}} E_{X_\sigma} f(X_t).
\]

Let \( A \in \mathcal{F}_\sigma \) be arbitrary. By the previous \( A \cap \{ \sigma = s \} \in \mathcal{F}_\sigma \). Use that \( A \cap \{ \sigma < \infty \} = \bigcup_{s \in S} (A \cap \{ \sigma = s \}) \) and linearity of expectations.

QED

**Corollary 3.3.3** Any discrete time Markov chain has the strong Markov property.

**Theorem 3.3.4** Let \( X \) be a an \((E, \mathcal{E})\)-valued Markov process adapted to the filtration \((\mathcal{F}_t)_t\) with \( E \) a Polish space and \( \mathcal{E} \) the Borel-\( \sigma \)-algebra, and with right-continuous paths. Suppose that \( x \mapsto E_x f(X_s) = P_s f(x) \) is continuous, or, more generally, that \( t \mapsto E_{X_t} f(X_s) \) is right-continuous (everywhere) for each bounded continuous function \( f \). Then the strong Markov property holds.

**Proof.** Let \( \sigma \) be an \((\mathcal{F}_t)_t\)-stopping time. We will first show the strong Markov property for \( f \in b\mathcal{E} \), continuous. Then for indicator functions \( 1_{\{ B \}} \), \( B \in \mathcal{E} \), with \( B \) closed, by an approximation argument. Since the closed sets form a \( \pi \)-system for \( \mathcal{E} \), the \( d \)-system recipe provides us the strong Markov property for all of \( \mathcal{E} \). The standard machinery finally provides us the result for \( f \in b\mathcal{E} \).

Let first \( f \) be a bounded and continuous function, let \( t \geq 0 \). \( \mathcal{F}_\sigma \)-measurability of \( 1_{\{ \sigma < \infty \}} X_\sigma \) and \( 1_{\{ \sigma < \infty \}} E_{X_\sigma} f(X_t) \) can be checked analogously to the previous lemma.

Consider

\[
\sigma_m = \sum_{k=1}^\infty \frac{1}{2^m} \cdot 1_{\{ \frac{k-1}{2^m} < \sigma \leq \frac{k}{2^m} \}} + \infty \cdot 1_{\{ \sigma = \infty \}}.
\]

Then \( \sigma_m \) takes countably many different values and \( \sigma_m \downarrow \sigma \). By virtue of Lemma 3.3.2 for all \( A \in \mathcal{F}_{\sigma_m} \)

\[
E_\nu 1_{\{ \sigma_m < \infty \}} f(X_{\sigma_m + t}) = E_\nu 1_{\{ \sigma_m < \infty \}} E_{X_{\sigma_m}} f(X_t).
\]

Next, use that if \( A \in \mathcal{F}_\sigma \), then \( A \in \mathcal{F}_{\sigma_m} \). Moreover,

\[
1_{\{ A \}} 1_{\{ \sigma_m < \infty \}} f(X_{\sigma_m + t}) \to 1_{\{ A \}} 1_{\{ \sigma < \infty \}} f(X_{\sigma + t})
\]
3.3.

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and

\[ 1_{\{A\}}1_{\{\sigma_m < \infty\}}E_{X_{\sigma_m}}f(X_t) \rightarrow 1_{\{A\}}1_{\{\sigma < \infty\}}E_{X_{\sigma}}f(X_t), \quad m \rightarrow \infty. \]

Apply dominated convergence.

Next we will show that the strong Markov property holds for \(1_{\{B\}}, B \in \mathcal{E}\). Let \(B \in \mathcal{E}\) be a closed set and let \(f^m\) be given by

\[ f^m(x) = 1 - m \cdot (m^{-1} \wedge d(x, B)), \quad m = 1, \ldots, \]

where \(d\) is a metric on \(E\), consistent with the topology. Then \(f^m \in b\mathcal{E}\) is continuous and by the previous

\[ E1_{\{A\}}1_{\{\sigma < \infty\}}f^m(X_t) \circ \theta_\sigma = E1_{\{A\}}1_{\{\sigma < \infty\}}E_{X_\sigma}f^m(X_t), \quad A \in \mathcal{F}_\sigma. \]

The random variable on the left-hand side converges pointwise to \(1_{\{A\}}1_{\{\sigma < \infty\}}1_{\{B\}} \circ \theta_\sigma\), the one on right-hand side converges pointwise to \(1_{\{A\}}1_{\{\sigma < \infty\}}E_{X_\sigma}1_{\{B\}}\). Use monotone convergence.

QED

**Corollary 3.3.5**
Assume that \(X\) is a right-continuous process with a countable state space \(E\), equipped with the discrete topology, and \(\mathcal{E} = 2^E\). Then \(X\) has the strong Markov property.

The corollary implies that the Poisson process has the strong Markov property, as well as the right-continuous Markov jump process.

**Corollary 3.3.6**
BM, the Ornstein-Uhlenbeck process, and geometric BM have the strong Markov property.

Without the required continuity properties, the strong Markov property may fail, as illustrated in Example 3.4.1. We discuss some other aspects of the strong Markov property in connection with optional times.

3.3.2 Intermezzo on optional times: Markov property and strong Markov property

Sometimes (eg. the book by Karatzas and Shreve) the strong Markov property is defined through optional times. In other words, the strong Markov property is defined to hold if for every function \(f \in b\mathcal{E}\), any adapted optional time \(\sigma\) and any initial distribution \(\nu\) and any \(t \geq 0\)

\[ 1_{\{\sigma < \infty\}}E_\nu(f(X_{\sigma+t}) | \mathcal{F}_{\sigma+}) = 1_{\{\sigma < \infty\}}E_{X_\sigma}f(X_t), \quad \mathbb{P}_\nu \text{ a.s.} \quad (3.3.2) \]

The conditions under which this alternative strong Markov property holds, are analogous to the conditions guaranteeing the strong Markov property (3.3.1) to hold by Lemma 1.6.7 (see Chapter 1 the new observation on the \(\sigma\)-algebra \(\mathcal{F}_{\tau^+}\) and its characterisations in Exercises 1.34 and 1.35). In this case, you have to replace \(\mathcal{F}_\sigma\) etc. by \(\mathcal{F}_{\sigma^+}\) in the above statements. However, one needs to show that the Markov property holds with respect to the filtration \((\mathcal{F}_{\tau^+})_t\). This holds true under the conditions of Theorem 3.3.4.

**Corollary 3.3.7**
Assume the conditions of Theorem 3.3.4.

i) Then \(X\) is Markov with respect to \((\mathcal{F}_{\tau^+})_t\).
ii) Let $\tau$ be an $(F_t)_{t}$-optional time. Then (3.3.2) holds for any function $f \in bE$.

iii) Let $\tau$ be a finite adapted optional time. Let $Y$ be a bounded $F_{\tau^+}$-measurable random variable. Then

$$E_\nu Y f(X_{\tau^+}) = E_\nu(Y E_{X_\tau} f(X_t)).$$

Proof. See Exercise 3.10.

An interesting consequence is the following.

**Lemma 3.3.8** Assume the conditions of Theorem 3.3.4.

i) Blumenthal’s 0-1 Law If $A \in F_{0^+}$ then $P_x(A) = 0$ or 1 for all $x \in E$.

ii) If $\tau$ is an $(F_t^X)_{t}$-optional time, then $P_x\{\tau = 0\} = 0$ or 1, for all $x \in E$.

Proof. See Exercise 3.10.

The generalised Markov property w.r.t. $(F_t^X)_{t}$ is now an immediate consequence of Theorem 3.2.4, and the strong Markov property w.r.t. the same filtration is a consequence of Theorem 3.3.4.

Before moving to generalised version of the strong Markov property, we will state an important property characterising the so-called sojourn time in a state.

**Lemma 3.3.9** Let $X$ be a canonical, right-continuous Markov process with values in a Polish state space $E$, equipped with the Borel $\sigma$-algebra $E$. Assume that $t \mapsto E_{X_t} f(X_s)$ is right-continuous, for each bounded continuous function $f : E \to \mathbb{R}$. Define $S_x = \inf\{t > 0 | X_t \neq x\}$. This is the sojourn time in state $x$, when $X_0 = x$.

- Then for each $x \in E$, there exists $\alpha_x \in [0, \infty]$, such that
  $$P_x\{S_x > t\} = e^{-\alpha_x t}, \quad t \geq 0.$$  
  If $\alpha_x = 0$, then $x$ is called an absorption point or a trap. If $\alpha_x \in (0, \infty)$, then $x$ is called a holding point, and if $\alpha_x = \infty$, then $x$ is a regular point.

- Suppose that $x$ is a holding point. Then, there exists a probability distribution $q$ on $(E, E)$, such that
  $$P_x\{X_{S_x} \in B, t < S_x < \infty\} = e^{-\alpha_x t} q(B).$$

Proof. see Exercise 3.12!

**3.3.3 Generalised strong Markov property for right-continuous canonical Markov processes**

We are now interested in the question under what conditions the analogon of Theorem 3.2.4 holds. In order to formulate this analogon properly, we need introduce the concept of a random time shift. However, the use of time shifts restricts us to canonical processes.
3.3. **STRONG MARKOV PROPERTY**

Let $X$ be a canonical Markov process w.r.t the filtration $(\mathcal{F}_t)_{t \geq 0}$, where we again assume the set-up described in the section 3.2, prior to Corollary 3.2.2 and Theorem 3.2.4. Suppose that $X$ has everywhere right-continuous sample paths.

For a random time $\tau$ we now define $\theta_{\tau}$ as the operator that maps the path $s \mapsto \omega_s$ to the path $s \mapsto \omega_{\tau+\omega_s}$. If $\tau$ equals the deterministic time $t$, then $\tau(\omega) = t$ for all $\omega$ and so $\theta_{\tau}$ equals the old operator $\theta_t$.

Since the canonical process $X$ is just the identity on the space $\Omega$, we have for instance that $(X_t \circ \theta_{\tau})(\omega) = X_t(\theta_{\tau}(\omega)) = (\theta_{\tau}(\omega))_t = \omega_{\tau+\omega_s} = X_{\tau+s}$, in other words $X_t \circ \theta_{\tau} = X_{\tau+t}$. So the operators $\theta_{\tau}$ can still be viewed as time shifts.

The first results deals with countably valued stopping times.

**Lemma 3.3.10** Let $X$ be a canonical Markov process. Then

$$1_{\{\sigma < \infty\}} E_\nu(Z \circ \theta_\sigma | \mathcal{F}_\sigma^X) = 1_{\{\sigma < \infty\}} E_{X_\sigma} Z \quad \text{P}_\nu \text{ a.s.} \quad (3.3.3)$$

for any bounded or non-negative $\mathcal{F}_\sigma^X$-measurable random variable $Z$, any initial distribution $\nu$ and any stopping time $\sigma$, for which there exists a countable subset $S \subset [0, \infty)$ such that $\sigma \in S \cup \{\infty\}$.

**Proof.** The map $1_{\{\sigma < \infty\}} E_{X_\sigma} Z$ is $\mathcal{F}_\sigma^X$-measurable if

$$\{1_{\{\sigma < \infty\}} E_{X_\sigma} Z \in B\} \cap \{\sigma = t\} = \{E_{X_t} Z \in B\} \cap \{\sigma = t\} \in \mathcal{F}_t^X$$

for all $B \in \mathcal{B}, \ t \geq 0$. This follows from Lemma 3.2.3.

The next step is to show that

$$E_\nu 1_{\{A\}} 1_{\{\sigma < \infty\}} Z \circ \theta_\sigma = E_\nu 1_{\{A\}} 1_{\{\sigma < \infty\}} E_{X_\sigma} Z, \quad A \in \mathcal{F}_\sigma^X.$$  

This follows analogously to the similar assertion in the proof of Lemma 3.3.2, but with $Z \circ \theta_\sigma$ replacing $f(X_{\sigma+t})$. \hfill QED

As is the case with the strong Markov property, for general stopping times we need additional conditions.

**Theorem 3.3.11** Let $X$ be a an $(E, \mathcal{E})$-valued canonical Markov process with $E$ a Polish space and $\mathcal{E}$ the Borel-$\sigma$-algebra, and with right-continuous paths. Suppose that $x \mapsto E_x f(X_s) = P_s f(x)$ is continuous, or, more generally, $t \mapsto E_x f(X_s)$ is right-continuous everywhere for each bounded continuous function $f$. Then (3.3.3) holds for any bounded or non-negative $\mathcal{F}_\infty^X$-measurable random variable $Z$, any initial distribution $\nu$ and any stopping time $\sigma$.

**Proof.** Let $\sigma$ be an $(\mathcal{F}_t)_t$-adapted stopping time and $\nu$ an initial distribution. We will first prove the result for $Z = \prod_{n=1}^n 1_{\{A_n\}}(X_{t_n})$, with $n \in \mathbb{Z}_+, t_1 < \cdots < t_n, A_1, \ldots, A_n \in \mathcal{E}$ by an induction argument. For $n = 1$ the result follows from Theorem 3.3.4. For $n = 2$

$$E_\nu 1_{\{A_1\}}(X_{t_1}) 1_{\{A_2\}}(X_{t_2}) \circ \theta_\sigma | \mathcal{F}_\sigma = E_\nu 1_{\{A_1\}}(X_{\sigma+t_1}) 1_{\{A_2\}}(X_{\sigma+t_2}) | \mathcal{F}_\sigma$$

$$= E_\nu 1_{\{A_1\}}(X_{\sigma+t_1}) E_\nu 1_{\{A_2\}}(X_{\sigma+t_2}) | \mathcal{F}_{\sigma+t_1} | \mathcal{F}_\sigma$$

$$= E_\nu 1_{\{A_1\}}(X_{\sigma+t_1}) E_\nu 1_{\{A_2\}}(X_{\sigma+t_2}) | \theta_{\sigma+t_1} | \mathcal{F}_{\sigma+t_1} | \mathcal{F}_\sigma$$

$$= E_\nu 1_{\{A_1\}}(X_{\sigma+t_1}) E_{X_{\sigma+t_1}} 1_{\{A_2\}}(X_{t_2-t_1}) | \mathcal{F}_\sigma$$

$$= E_{X_\sigma} g(X_{t_1}),$$
where on the event \( X_{t_1} = x \)
\[
g(X_{t_1}) = g(x) = 1_{\{A_1\}}(x)E_x1_{\{A_2\}}(X_{t_2-t_1}) = 1_{\{A_1\}}P_{t_2-t_1}1_{\{A_2\}}(x),
\]
so that \( g(X_{t_1}) = 1_{\{A_1\}}P_{t_2-t_1}1_{\{A_2\}}(X_{t_1}) \). Further, on the event \( X_{\sigma} = y \)
\[
E_{X_{\sigma}}g(X_{t_1}) = E_y1_{\{A_1\}}P_{t_2-t_1}1_{\{A_2\}}(X_{t_1})
= P_t1_{\{A_1\}}P_{t_2-t_1}1_{\{A_2\}}(X_{t_1})(y)
= E_y1_{\{A_1\}}(X_{t_1})1_{\{A_2\}}(X_{t_2}).
\]
This yields
\[
E_{X_{\sigma}}g(X_{t_1}) = E_{X_{\sigma}}1_{\{A_1\}}(X_{t_1})1_{\{A_2\}}(X_{t_2}),
\]
which gives us the desired relation for \( n = 2 \). The general induction step follows similarly.

Finally, we apply the \( d \)-system recipe to show that the strong Markov property holds for \( Z = 1_{\{B\}} \) with \( B \in \mathcal{F}_X^\infty \). Then use the standard machinery. QED

The strong Markov property has interesting consequences for right-continuous canonical Markov processes \( X \) with so-called stationary and independent increments. This means that \( X_t - X_s \) is independent of \( \mathcal{F}_s \) for \( s \leq t \), and for each initial distribution \( \nu \), the \( \mathbb{P}_\nu \)-distribution of \( X_t - X_s \) only depends on the difference \( t - s \), and is independent of \( \nu \). In other words: the \( \mathbb{P}_\nu \)-distribution of \( X_t - X_s \) and the \( \mathbb{P}_\mu \) distribution \( X_{t-s} - X_0 \) are equal for all initial distributions \( \nu \) and \( \mu \), provided that \( E \) is an Abelian group.

The \( \text{Lévy processes} \) are a class of processes with this property of which canonical BM and the canonical Poisson process are well-known examples. In fact, one can prove that if \( X \) is a stochastic process satisfying the conditions of the lemma below, for \( E = \mathbb{R} \), then \( X \) is a Gaussian process!

**Lemma 3.3.12** Let \( E \) be a separable Banach space. Let \( X \) be a right-continuous process with values in \((E, \mathcal{B}(E))\), defined on \((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})\). Suppose that \( X \) has stationary, independent increments.

i) \( X \) is a Markov process with initial distribution \( \mathbb{P}\{X_0 \in \cdot\} \).

ii) Let \( \tau \) be a finite \((\mathcal{F}_t)\)-stopping time. Then the process \( X(\tau) = (X_{\tau+t} - X_{\tau})_{t \geq 0} \) is independent of \( \mathcal{F}_\tau \). It is a Markov process, adapted to the filtration \((\mathcal{F}_{\tau+t})_t \). The distribution \( \mathbb{P}_\tau \) of \( X(\tau) \) is the same as the distribution of \( X - X_0 \) under \( \mathbb{P}_0 \).

**Proof.** See Exercise 3.11. We give some hints for the proof of part (ii). Put \( Y_t = X_{\tau+t} - X_{\tau}, \) \( t \geq 0 \). For \( t_1 < \cdots < t_n \) and functions \( f_1, \ldots, f_n \in b\mathcal{E} \) we have
\[
E_\nu\left( \prod_k f_k(Y_{t_k}) \mid \mathcal{F}_\tau \right) = E_\nu\left( \prod_k f_k(X_{\tau+t_k} - X_{\tau}) \mid \mathcal{F}_\tau \right)
= E_{X_\tau}\prod_k f_k(X_{t_k} - X_0), \quad \mathbb{P}_\nu \text{ a.s.},
\]
by the strong Markov property. As a consequence, the proof is complete once we have shown that for arbitrary \( x \in E \)
\[
E_x\prod_{k=1}^n f_k(X_{t_k} - X_0) = P_{t_1}f_1 \cdots P_{t_n-t_{n-1}}f_n(0),
\]
(cf. Characterisation Lemma 3.1.5). Prove this by induction on $n$. 

QED

Surprisingly enough, the strong Markov property together with cadlag paths and bounded jumps, are sufficient to guarantee finiteness of all moments for a real-valued stochastic process with stationary and independent increments.

**Lemma 3.3.13** Assume the conditions of Lemma 3.3.12, and in addition that $X$ is a real-valued canonical stochastic process with cadlag paths, $t \mapsto E_{X_t} f(X_s)$ is right-continuous for each bounded continuous function $f$ and there exists a constant $C$ such that

$$\lim \inf_{s \uparrow t} X_s \leq X_t + C, \quad t \geq 0.$$ 

Then $E_x |X_t|^p < \infty$, $p = 1, 2 \ldots$, $t \geq 0$.

**Proof.** The proof can be based on the following steps.

- Show that for $f : [0, t] \to \mathbb{R}$ a cadlag function, it holds that $\sup_{s \in [0, t]} |f(s)| < \infty$.
- Let $n > C$. Let $X_0 = 0$, and let $\tau_n = \inf \{ s \mid |X_s| \geq n \}$. Note, that $\{ \tau_n \leq t \} = \{ \sup_{s \leq t} |X_s| \geq n \}$. Show, that there exist $N > 0$, such that

$$q := P_0 \left\{ \sup_{0 \leq s \leq t} |X_s| \geq N \right\} < 1.$$

- Using the strong Markov property, show, that

$$P_0 \{ \sup_{s \leq t} |X_s| \geq N + C + y \} \leq q \cdot P_0 \{ \sup_{0 \leq s \leq t} |X_s| \geq y \}.$$ 

QED

The following lemma is often useful in connection with the strong Markov property.

**Lemma 3.3.14** Let $\Omega = \Gamma, \mathcal{F} = \mathcal{E}^T \cap \Gamma$ with $\Gamma \subset E^T$ such that $\omega \in \Gamma$ implies (i) $\omega' = \theta_s(\omega) \in \Gamma$ for all $s \in T$ and (ii) for all $t \in T$, $\omega' = (\omega_{s\wedge t})_s \in \Gamma$. Let $X$ be the canonical process, adapted to the filtration $(\mathcal{F}_t^X)_t$.

i) If $\sigma$ and $\tau$ are finite $(\mathcal{F}_t^X)_t$-stopping times, then $\sigma + \tau \circ \theta_\sigma$ is also a finite $(\mathcal{F}_t^X)_t$-stopping time.

ii) If $\sigma$ and $\tau$ are finite $(\mathcal{F}_t^+)_t$-optional times, then $\sigma + \tau \circ \theta_\sigma$ is also a finite $(\mathcal{F}_t^+)_t$-optional time.

**Proof.** We will prove (i). By Lemma 1.6.5 (b) Galmarino test we have to show that (i) $(\sigma + \tau \circ \theta_\sigma)$ is $\mathcal{F}_\infty^X$-measurable and (ii) for all $t \geq 0$, $\sigma(\omega) + \tau \circ \theta_\sigma(\omega) \leq t$ and $\omega_s = \omega'_s$, $s \leq t$, implies that $\sigma(\omega') + \tau \circ \theta_\sigma(\omega') \leq t$.

The first statement is simply proved, for instance by a similar reasoning to Exercise 3.14. Let us consider the second statement.

For $t = 0$, $\sigma(\omega) + \tau \circ \theta_\sigma(\omega) \leq 0$ then $\sigma(\omega), \tau(\omega) \leq 0$. Let $\omega'_0 = \omega_0$. Since $\sigma$ is a stopping time, the Galmarino test applied to $\sigma$ and $\tau$ implies that $\sigma(\omega'), \tau(\omega') \leq 0$, hence $\sigma(\omega') + \tau \circ \theta_\sigma(\omega') \leq 0$. 


Next let $t > 0$. Let $\omega'$ satisfy $\omega'_s = \omega_s$, $s \leq t$. Then $\sigma(\omega) \leq t$, and by the above and the Galmarino test, $\sigma(\omega') \leq t$.

Is it possible that $\sigma(\omega) < \sigma(\omega')$? Suppose this is the case. Then there exists $s < t$ such that $\sigma(\omega) \leq s < \sigma(\omega')$. But by the Galmarino test, $\sigma(\omega') \leq s$. A contradiction. Hence $\sigma(\omega') \leq \sigma(\omega)$. A repetition of the same argument with the roles of $\omega$ and $\omega'$ interchanged, yields that in fact $\sigma(\omega) = \sigma(\omega')$.

But then $\tau \circ \theta_\sigma(\omega) = \tau \circ \theta_\sigma(\omega')$. Hence $t \geq \sigma(\omega) + \tau \circ \theta_\sigma(\omega) = \sigma(\omega') + \tau \circ \theta_\sigma(\omega')$. This is what we had to prove.

The proof of (ii) is Exercise 3.14.

QED

3.4 Applications to Brownian Motion

3.4.1 Reflection principle

The first example that we give, is the so-called reflection principle (compare with Ch. 1, Exercise 1.12). First note that Lemma 3.3.12 implies that $(W_{\tau+t} - W_\tau)_t$ is BM, provided $\tau$ is an everywhere finite stopping time.

Recall that we denote the hitting time of $x \in \mathbb{R}$ by $\tau_x$. This is an a.s. finite stopping time with respect to the natural filtration of the BM (see Example 1.6.1). The problem is that $\tau_x$ is not necessarily finite everywhere and hence we cannot conclude that $(W_{\tau_x+t} - W_{\tau_x})_t$ is a BM, since it need not be defined everywhere.

The solution is to restrict to a smaller underlying space, but this trick might cause problems, if we need consider different initial distributions (a null-set under one initial distribution need not necessarily be a null-set under different initial distribution...). The simplest approach is via approximations: $(W_{\tau_x+n} - W_{\tau_x})_t$ is a BM, for each $n \geq 0$.

**Theorem 3.4.1 (Reflection principle)** Let $W$ be a Brownian motion with continuous paths. Let $x \in \mathbb{R}$ be given. Define the process $W'$ by

$$W'_t = \begin{cases} W_t, & t \leq \tau_x \leq \infty \\ 2x - W_t, & t > \tau_x. \end{cases}$$

Then $W'$ is a standard BM with continuous paths.

**Proof.** If $x = 0$, $\tau_0 = 0$, and so the assertion is equivalent to symmetry of BM.

Let $x \neq 0$. Define processes $Y^n$ and $Z^n$ by $Y = W_{\tau_x+n}$ and $Z^n = W_{\tau_x+n+t} - W_{\tau_x+n}$, $t \geq 0$. By Theorem 3.3.12 the processes $Y^n$ and $Z^n$ are independent and $Z^n$ is a standard BM. By symmetry of BM, it follows that $-Z^n$ is also a BM that is independent of $Y^n$, and so the two pairs $(Y^n, Z^n)$ and $(Y^n, -Z^n)$ have the same distribution (i.e. the fdd’s are equal).

Now, for $t \geq 0$

$$W_t = Y^n_t + Z^n_{t-\tau_x+n}1_{\{t>\tau_x\}}.$$ 

Define for $t \geq 0$

$$W'^n_t = Y^n_t - Z^n_{t-\tau_x+n}1_{\{t>\tau_x\}}.$$ 

The $W'^n_t$ is $W$ reflected about the value $W_{\tau_x+n}$. By the continuity of Brownian motion paths, $W'^n_t \to W'_t$, $n \to \infty$ a.s., for all $t \geq 0$. 

Write $C_0[0,\infty) = \{ \omega \in C[0,\infty) \mid \omega_0 = 0 \}$. We have $W = \phi^n(Y^n, Z^n)$ and $W'^n = \phi^n(Y^n, -Z^n)$, where $\phi^n : C[0,\infty) \times C_0[0,\infty) \to C[0,\infty)$ is given by
\[
\phi^n(y, z)(t) = y(t) + z(t - \psi^n(y))1_{\{t > \psi(y)\}},
\]
where $\psi^n : C[0,\infty) \to [0,\infty]$ is defined by $\psi^n(y) = n \wedge \inf\{t > 0 \mid y(t) = x\}$. Consider the induced $\sigma$-algebra on $C[0,\infty)$ and $C_0[0,\infty)$ inherited from the $\sigma$-algebra $\mathcal{B}^{(0,\infty)}$ on $\mathbb{R}^{(0,\infty)}$. It is easily verified that $\psi^n$ is a Borel-measurable map, and that $\phi^n$ is measurable as the composition of measurable maps (cf. Exercise 3.17). Since $(Y^n, Z^n) \overset{d}{=} (Y^n, -Z^n)$, it follows that $W = \phi^n(Y^n, Z^n) \overset{d}{=} \phi^n(Y^n, -Z^n) = W'^n$.

On the time-interval $[0,n]$ clearly $W'^m$ and $W'$ have equal paths for $m > n$, and so they have the same fdd on $[0,n]$. This implies that the fdd of $W'$ are multivariate Gaussian with the right mean and covariance functions. Since $W'$ has continuous paths, we can invoke Lemma 1.4.3 to conclude that $W'$ is BM with everywhere continuous paths. Note that the constructed process is not canonical. QED

The reflection principle allows us to calculate the distributions of certain functionals related to the hitting times of BM. We first consider the joint distribution of $W_t$ and the running maximum
\[
S_t = \sup_{s \leq t} W_s.
\]

**Corollary 3.4.2** Let $W$ be a standard BM and $S$ its running maximum. Then
\[
P\{W_t \leq x, S_t \geq y\} = P\{W_t \leq x - 2y\}, \quad x \leq y.
\]
The pair $(W_t, S_t)$ has joint density
\[
(x, y) \mapsto \frac{(2y - x)e^{-(2y-x)^2/2t}}{\sqrt{\pi t}^3/2} 1_{\{x \leq y\}},
\]
with respect to the Lebesgue measure.

**Proof.** Let first $y > 0$. Let $W'$ be the process obtained by reflecting $W$ at the hitting time $\tau_y$. Observe that $S_t \geq y$ if and only if $t \geq \tau_y$. Hence, the probability of interest equals $P\{W_t \leq x, t \geq \tau_y\}$. On the event $\{t \geq \tau_y\}$ we have $W_t = 2y - W'_t$, and so we have to calculate $P\{W'_t \geq 2y - x, t \geq \tau_y\}$. Since $x \leq y$, we have $2y - x \geq y$. hence $\{W'_t \geq 2y - x\} \subseteq \{W'_t \geq y\} \subseteq \{t \geq \tau_y\}$. It follows that $P\{W'_t \geq 2y - x, t \geq \tau_y\} = P\{W'_t \geq 2y - x\}$. By the reflection principle and symmetry of BM this proves the first statement for $y > 0$.

For $y = 0$, we have $\tau_y = 0$, and $W'_t = -W_t$, showing the first statement directly.

The second statement follows from Exercise 3.18. QED

It follows from the preceding corollary that for all $x > 0$ and $t \geq 0$,
\[
P\{S_t \geq x\} = P\{\tau_x \leq t\} = 2P\{W_t \geq x\} = P\{|W_t| \geq x\}
\]
(see Exercise 3.18). This shows in particular that $S_t \overset{d}{=} |W_t|$ for every $t \geq 0$. This allows to construct an example of a Markov process that lacks the strong Markov property.
Example 3.4.1 (Strong Markov property fails (Yushkevich)) Consider Example 3.1.2 (A). Instead of the canonical process $X$:

$$X_t = X_0 + W_t,$$

with $X_0 \overset{d}{=} \nu$, independent of $W$, we consider

$$\tilde{X}_t = X_0 + 1_{\{X_0 \neq 0\}}W_t = 1_{\{X_0 \neq 0\}}X_t.$$

For initial distribution $\nu = \delta_x$, $x \neq 0$, the underlying distributions of $X$ and $\tilde{X}$ are equal.

\(\tilde{X}\) is a Markov process with respect to the same filtration and with transition function

$$\tilde{P}_t(x,B) = \begin{cases} \int_B \frac{1}{\sqrt{2\pi t}} e^{-(y-x)^2/2t} dy, & x \neq 0 \\ \delta_x(B), & x = 0. \end{cases}$$

Suppose that $\tilde{X}$ has the strong Markov property. Let $\tau = \inf\{t \geq 0 \mid \tilde{X}_t = 0\} = \inf\{t \geq 0 \mid X_t = 0\}$. It is a stopping time for both $\tilde{X}$ and $X$. Consider the function $f = 1_{\{R \setminus \{0\}\}}$. Clearly $f(X_1) = f(\tilde{X}_1)$ on $1_{\{X_0 \neq 0\}}$. Then,

$$1_{\{\tau < \infty\}}E_{\tilde{X}_0} f(\tilde{X}_1) = 0,$$

(3.4.1)

by definition. Take initial distribution $\nu = \delta_x$ for some $x > 0$, and choose $A = 1_{\{t \leq 1\}}$. Then, $A \in \mathcal{F}_\tau$ and $\tau < \infty$ on $A$. By (3.4.1), $E_x 1_A E_{\tilde{X}_0} f(X_1) = 0$. However,

$$E_x 1_A f(\tilde{X}_{\tau+1}) = E 1_A f(X_{\tau+1}) = E_x 1_A E_{\tilde{X}_0} f(X_1) = E_x 1_{\{\tau \leq 1\}}P_0\{X_1 \neq 0\} = P\{\tau \leq 1\} = P\{|W_1| \geq x\} > 0,$$

a contradiction with the strong Markov property. The details have to be worked out in Exercise 3.16.

3.4.2 Ratio limit result

In this subsection $W$ is the canonical BM on $(\Omega = C[0, \infty), \mathcal{F} = C(0, \infty) \cap B^{\mathbb{R}^+})$ with associated Markov process $X$ (cf. Example 3.1.2). Since BM has stationary, independent increments, Corollary 3.3.12 implies that for every $(\mathcal{F}_t)_t$-stopping time $\tau$, the proces $(X_{\tau+t} - X_{\tau})_t$ is a BM. This can be used to prove an interesting ratio limit result (originally derived by Cyrus Derman 1954).

To this end, let $A \in \mathcal{B}$ be a bounded set. Define $\mu(A, \tau) = \lambda\{t \leq \tau : X_t \in A\}$, where $\lambda$ is the Lebesgue measure (on $\mathbb{R}, \mathcal{B}$) and $\tau$ a finite $(\mathcal{F}_t)_t$-stopping time, w.r.t $P_0$. Denote $\tau_1 = \inf\{t > 0 \mid X_t = 1\}$, and (by abuse of previously introduced notation) $\tau_0 = \inf\{t > 0 \mid t \geq \tau_1, X_t = 0\}$.

**Lemma 3.4.3**

i) $\mu(A, \tau_1)$ is a measurable function on $(\Omega, \mathcal{F})$.

ii) $\mu(A) := E_0 \mu(A, \tau_1) = 2 \int_0^1 1_{\{A\}}(x) \lambda(x) + 2 \int_0^1 (1 - x) 1_{\{A\}}(x) d\lambda(x)$.

iii) $\mu'(A) := E_0 \mu(A, \tau_0) = 2\lambda(A)$. 
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Proof. See exercise 3.15. For the proof of (i), it is sufficient to show that (explain)

\((s, \omega) \mapsto 1_{\{A\}}(X_s(\omega))1_{([0,\tau(\omega)]]}(s)\)

is \(\mathcal{B}[0, \infty] \times \mathcal{F}/\mathcal{B}\)-measurable. To this end, show that \((Y_s(\omega) = 1_{\{0,\tau(\omega)]}(s))_{s \geq 0}\) is an \(\mathcal{F}/\mathcal{B}\)-progressively measurable stochastic process.

For the proof of (ii), note that

\[E_0 \mu(A, \tau_1) = E_0 \int_0^\infty 1_{\{A\}}(X_t)1_{([t,\infty))}(\tau_1) \, d\lambda(t) = \int_\Omega \int_0^\infty 1_{\{A\}}(X_t)1_{([t,\infty))}(\tau_1) \, dt \, dP_0\]

\[= \int_0^\infty \int_\Omega 1_{\{A\}}(X_t)1_{([t,\infty))}(\tau_1) \, dP_0 \, dt\]

\[= \int_0^\infty P_0\{X_t \in A, t < \tau_1 \} \, dt\]

\[= \int_0^\infty \int_A w(t, x) \, d\lambda(x) \, dt\]

\[= \int_A \int_0^\infty w(t, x) \, dt \, d\lambda(x),\]

where

\[w(t, x) = \begin{cases} \frac{1}{\sqrt{2\pi t}} (e^{-x^2/2t} - e^{-(x-2)^2/2t}), & x \leq 1 \\ 0, & x > 1. \end{cases}\]

This follows from

\[P_0\{X_t \in A \cap [-\infty, 1], \tau_1 < t\} = P_2\{X_t \in A \cap [-\infty, 1]\} .\]

Why is this true? (ii) can then be shown by writing

\[w(t, x) = -1_{\{(-\infty, 1]\}}(x) \int_{x-2}^x \frac{u}{t^{3/2} \sqrt{2\pi}} e^{-u^2/2t} \, du,\]

applying Fubini, and doing a substitution \(s = t^{-1/2}\). Distinguish the cases that \(x \leq 0\) and \(0 < x \leq 1\). QED

Let \(f, g : \mathbb{R} \to \mathbb{R}\) be Lebesgue measurable, integrable functions with \(\int_\mathbb{R} g(x) \, d\lambda(x) \neq 0\).

**Theorem 3.4.4**

\[\lim_{T \to \infty} \frac{\int_0^T f(W_t) \, dt}{\int_0^T g(W_s) \, ds} = \frac{\int_\mathbb{R} f(x) \, d\lambda(x)}{\int_\mathbb{R} g(x) \, d\lambda(x)}, \quad \text{a.s.}\]

**Proof.** Put \(\tau_0^1 = \tau_0\) and \(\tau_1^1 = \tau_1\). Inductively define for \(n \geq 2\): \(\tau_0^n = \inf\{t \geq \tau_0^n | X_t = 0\}\), and \(\tau_1^n = \inf\{t \geq \tau_0^{n-1} | X_t = 1\}\). By virtue of the standard machinery, one has

\[E_0 \int_0^{\tau_0^1} f(X_t) \, dt = 2 \int_\mathbb{R} f(x) \, dx.\]

Now, for any \(T \geq 0\) define

\[K(T) = \max\{n | \tau_0^n \leq T\}.\]

Then \(\lim_{T \to \infty} \frac{\int_0^T f(X_t) \, dt}{K(T)} = 2 \int_\mathbb{R} f(x) \, dx, \ P_0\text{-a.s.} \). The result then follows. QED
3.4.3 Hitting time distribution

We also may derive an explicit expression for the density of the hitting time $\tau_x$. It is easily seen from this expression that $E\tau_x = \infty$, as was proved by martingale methods in Exercise 2.30 of Chapter 2.

**Corollary 3.4.5** The first time $\tau_x$ that the standard BM hits the level $x > 0$ has density

$$ t \mapsto \frac{xe^{-x^2/2t}}{\sqrt{2\pi t^3}}1\{t \geq 0\}, $$

with respect to the Lebesgue measure.

*Proof.* See Exercise 3.19. \(\Box\)

We have seen in the first two Chapters that the zero set of standard BM is a.s. closed, unbounded, has Lebesgue measure zero and that 0 is an accumulation point of the set, i.e. 0 is not an isolated point. Using the strong Markov property we can prove that in fact the zero set contains no isolated point at all.

**Corollary 3.4.6** The zero set $Z = \{t \geq 0 \mid W_t = 0\}$ of standard BM is a.s. closed, unbounded, contains no isolated points and has Lebesgue measure 0.

*Proof.* In view of Exercise 1.29, we only have to prove that $Z$ contains no isolated points. For rational $q \geq 0$, define $\sigma_q = q + q_0 \circ \theta_q$. Hence, $\sigma_q$ is the first time after (or at) time $q$ that BM visits 0. By Lemma 3.3.14 the random time $\sigma_q$ is an optional time. The strong Markov property implies that $W_{\sigma_q+t} - W_{\sigma_q}$ is a BM w.r.t $(\mathcal{F}_{\tau_q}^X)$. By Corollary 2.4.6 it follows that $\sigma_q$ a.s. is an accumulation point of $Z$. Hence, with probability 1 it holds that for every rational $q \geq 0$, $\sigma_q$ is an accumulation point of $Z$. Now take an arbitrary point $t \in Z$ and choose rational points $q_n$ such that $q_n \uparrow t$. Since $q_n \leq \sigma_{q_n} \leq t$, we have $\sigma_{q_n} \to t$. The limit of accumulation points is an accumulation point. This completes the proof. \(\Box\)

3.4.4 Embedding a random variable in Brownian motion

This subsection discusses the set-up of a result by Skorokhod, that a discrete martingale $M = (M_n)_{n=0,...}$ with independent increments can be ‘embedded’ in Brownian motion. Dubins\(^2\) derived a more general result with an elegant proof, part of which we will discuss next.

We consider standard BM. The filtration to be considered is the natural filtration $(\mathcal{F}_t^W)$ generated by BM itself.

Let $X$ be an integrable random variable (with values in $(\mathbb{R}, \mathcal{B})$) on an underlying probability space. Let $\mu$ denote the induced distribution of $X$ on $(\mathbb{R}, \mathcal{B})$. Denote $EX = m$.

Our aim construct an $(\mathcal{F}_t^W)$-stopping time $\tau$, such that $m + W_\tau$ has distribution $\mu$, i.e. $m + W_\tau \overset{d}{=} X$. The method is based on constructing a suitable Doob martingale with limiting distribution $\mu$ and then duplicate the same construction within BM.

3.4. APPLICATIONS TO BROWNIAN MOTION

The martingale that does the trick  We are going to associate a Doob martingale $M = (M_n)_{n=0,1,...}$ with $X$ as follows. At time $n$ construct inductively:

- a partition of $\mathbb{R}$ into a finite number of sets, denoted $G_n$;
- a set of points $S_n$ with $|S_n| = |G_n|$ and $|S_n \cap A| = 1$ for all $A \in G_n$;
- a $\sigma$-algebra $G_n \subset \mathcal{F}$.

We first prescribe initial values, and then the iterative construction.

For $n = 0$: $G_0 = \{\mathbb{R}\}$, $\mathcal{G}_0 = \{\emptyset, \Omega\}$, $S_0 = \{m_0\}$. For all $A \in G_n$, let $x^n_A = S_n \cap A$. The partition $G_{n+1}$ is generated from $G_n$ as follows. Let $A \in G_n$.

- If $A \in G_n$ and $\mathbb{P}\{X^{-1}\{x^n_A\}\} = \mathbb{P}\{X^{-1}(A)\}$, then $A \in G_{n+1}$;
- if $\mathbb{P}\{X^{-1}(\{x^n_A\}\} < \mathbb{P}\{X^{-1}(A)\}$ then $A \cap (-\infty, x^n_A], A \cap (x^n_A, \infty) \in G_{n+1}$.

Then put

$$G_{n+1} = \sigma(\{X^{-1}(A), A \in G_{n+1}\}).$$

and

$$S_{n+1} = \{x \in \mathbb{R} \mid \exists \omega \in \Omega \text{ such that } x = \mathbb{E}(X \mid G_{n+1})(\omega)\},$$

This construction defines a filtration $(G_n)_n \subset \mathcal{F}$ and a Doob martingale $(M_n = \mathbb{E}(X \mid \mathcal{G}_n))_n$. By the Levy upward theorem 2.2.15 $\mathbb{E}(X \mid \mathcal{G}_n) \to \mathbb{E}(X \mid \mathcal{G}_\infty)$ a.s. and in $L^1$.

Lemma 3.4.7 $X \overset{a.s.}{=} \mathbb{E}(X \mid \mathcal{G}_\infty)$.

For the proof, see the additional exercise.

As a consequence, the constructed Doob martingale determines the distribution of a random variable uniquely. By interpreting the sequence of realised values as a branching process, built up through a binary tree, it becomes easier to understand the embedding construction.

Binary tree  One can also construct an infinite binary tree associated with the above procedure: the nodes are pairs $(n, x^n_A)$, where $x^n_A \in S_n$, $n \geq 0$. Put an arrow $(n, x^n_A) \to (n + 1, x^n_B)$, for $A \in G_n$ and $B \in G_{n+1}$, with $B \subseteq A$. Then (cf. exercise) if two random variables having the same binary tree, are equal in distribution. Next, define

$$\mathcal{T}_n = (\mathbb{E}(X \mid G_0), \ldots, \mathbb{E}(X \mid G_n)),$$

the (finite) collection of realisations of the martingale $(M_k)_k$ upto time $n$, or alternatively the values along all paths of the root $m$ to level $n$.

Embedding of $X$  For $\omega \in \Omega$ let

$$\tau(\omega) = \inf\{t \geq 0 \mid \forall n = 0, 1, 2, \ldots \exists t_0 = 0 \leq \cdots \leq t_n \leq t, \text{ s. t. } (W_{t_0}(\omega), \ldots, W_{t_n}(\omega)) \in \mathcal{T}_n - m\},$$

where $\mathcal{T}_n - m$ means that $m$ is substracted from the values along the binary tree, we called these the ‘reduced’ values. In the binary tree, $(W_{t_0}(\omega), \ldots, W_{t_n}(\omega))$ is a sequence of reduced
values along paths in the binary tree from the root \(m\) with end-points in level \(n\). We further define random times \(\tau_n\) iteratively by putting \(\tau_0 = 0\) and

\[
\tau_{n+1}(\omega) = \inf\{t \geq \tau_n(\omega) \mid W_{\tau_{n+1}}(\omega) \in S_{n+1} - m\}.
\]

It is easily verified that \(\tau_n, n \geq 0\) and \(\tau\) are a.s. finite stopping times and that \(\{\tau_n\}_n\) is a non-decreasing sequence converging to a.s. to \(\tau\) (if \(\tau_n(\omega) = \infty\), we put \(\tau_n(\omega) = \infty\), for all \(m \geq n\)). Use unboundedness properties of BM paths, the continuity and, when viewing BM as a canonical process with the natural filtration, the Galmarino test. Note that \(\tau_0 = 0\)! As a consequence, \(W_{\tau_n} \to W_\tau\), a.s.

Next, let \(s^n_m = \max\{x \mid x \in S_n\}\) and \(s^n_m = \min\{x \mid x \in S_n\}\). Then \(\tau_n \leq \tau_{n+1} - m \land \tau^n_{n+1} - m\), the latter having finite expectation. As a consequence \(W_{\tau_n} = (W_{\tau_n})_t\) is a bounded martingale. Hence it is \(\mathcal{U}_1\). By virtue of Theorem 2.2.14, \(\mathbb{E}W_{\tau_n} = \mathbb{E}W_0 = 0\). Applying the optional sampling theorem 2.3.12 with stopping times \(\tau_n\) and \(\infty\) to \(W^{\tau_n+1}\) yields

\[
\mathbb{E}(W^{\tau_n+1} \mid \mathcal{F}_{\tau_n}) = W^{\tau_n+1}, \quad \text{a.s.}
\]

i.o.w.

\[
\mathbb{E}(W_{\tau_{n+1}} \mid \mathcal{F}_{\tau_n}) = W_{\tau_{n+1}}, \quad \text{a.s.}
\]

By the tower property, it follows that \(\mathbb{E}(W_{\tau_{n+1}} \mid \mathcal{F}_n) = W_{\tau_n}\), for \(\mathcal{F}_n = \sigma(W_{\tau_n})\). By continuity of BM paths, \(W_{\tau_n}(\omega)\) and \(W_{\tau_{n+1}}(\omega)\) have subsequent values along a path in the binary tree at levels \(n\) and \(n+1\). This implies that \(\mathcal{F}_n \subset \mathcal{F}_{n+1}\). Therefore, \(\{W_{\tau_n}\}_n\) is an \((\mathcal{F}_n)_n\)-martingale.

**Theorem 3.4.8**

i) \(\{W_{\tau_n}\}_n, W_\tau\) is an \((\mathcal{F}_n)_n\)-martingale with \(W_{\tau_n} \overset{\text{a.s.}}{\to} W_\tau\).

ii) \(m + W_\tau \overset{d}{=} X\).

iii) \(\mathbb{E} = \mathbb{E}X^2\).

Under the assumption that \(X\) has finite variance, the theorem is known as *Skorokhod’s first embedding theorem*. The present proof does not require this.

**Proof.** We first assume that \(X \geq 0\) and show \(\mathbb{E}W_\tau = 0\). By a.s. unboundedness of BM paths, \(W_{\tau_n} \to W_\tau\) a.s. Since \(\{W_{\tau_n}\}_n\) is a martingale bounded below by \(-m\), with \(\mathbb{E}|m + W_{\tau_n}| = \mathbb{E}(m + W_{\tau_n}) = m\), it is bounded in \(L^1\) and \(W_\tau\) is integrable.

Let \(s^n_m = \max\{x \mid x \in S_n\}\). If \(\sup_n s^n_m < \infty\), then \(\{W_{\tau_n}\}_n\) is a bounded martingale, hence \(\mathcal{U}_1\). It follows that \(W_\tau\) can be appended to the martingale sequence, and \(\mathbb{E}W_\tau = 0\).

Suppose that \(\sup_n s^n_m = \infty\). Let \(s^n = \max\{x \neq s^n_m \mid x \in S_n\}\), be the next largest value of \(S_n\). From the binary tree we may infer for \(\omega\) with \(W_{\tau_n(\omega)}(\omega) \leq s^n - m\), that \(W_{\tau_n(\omega)}(\omega) \leq s^n - m\) for all \(m \geq n\).

Consider the process \(Y_1 = W_{\tau_n}1_{\{(-m,s^n_m - m]\}}(W_{\tau_n})\), \(l \geq n\). This is an \((\mathcal{F}_l)_{l \geq n}\) bounded martingale. Indeed,

\[
\mathbb{E}(Y_{l+1} \mid \mathcal{F}_l) = 1_{\{(-m,s^n_m - m]\}}(W_{\tau_n})\mathbb{E}(W_{\tau_{l+1}} \mid \mathcal{F}_l) = 1_{\{(-m,s^n_m - m]\}}(W_{\tau_n})W_{\tau_{l+1}}.
\]

Hence, \(\{Y_l\}_{l \geq n}\) converges a.s. and in \(L^1\) to \(1_{\{(-m,s^n_m - m]\}}(W_{\tau_n})W_{\tau}\), and

\[
\mathbb{E}(1_{\{(-m,s^n_m - m]\}}(W_{\tau_n})W_{\tau} \mid \mathcal{F}_n) = 1_{\{(-m,s^n_m - m]\}}(W_{\tau_n})W_{\tau_n}.
\]
Taking expectations on both sides

\[ EW_\tau = -(s_n^u - m)P\{W_\tau_n = s_n^u - m\} + E1_{(s_n^u - m, \infty)}(W_\tau_n)W_\tau. \] (3.4.2)

Note that \( \{W_\tau_n\}_n \) being a martingale implies \( P\{W_\tau_n = x\} = P\{X \in A_n\} \), where \( x \in S_n \cap A_n - m \), for some \( A_n \in G_n \). This yields

\[ (s_n^u - m)P\{W_\tau_n = s_n^u - m\} = E\{(X - m)1_{([s_n^u - 1, \infty))}\} \to 0, \text{ as } n \to \infty. \] (3.4.3)

Furthermore, from the binary tree

\[ E1_{(s_n^u - m, \infty)}(W_\tau_n)W_\tau = E1_{(s_n^u - m, \infty)}(W_\tau)W_\tau \to 0, \text{ } n \to \infty. \] (3.4.4)

Combining (3.4.2), (3.4.3) and (3.4.4) yields \( EW_\tau = 0 \).

By Theorem 2.3.15 \( W_\tau \) can be appended to \( \{W_\tau_n\}_n \) to yield a supermartingale. That is, \( E(W_\tau | F_n) \leq W_\tau_n \). Taking expectations yields equality, hence the extended sequence is a martingale, in particular a Doob martingale. In other words \( W_\tau_n = E(W_\tau | F_n) \). The general case that \( X \) is not necessarily non-negative follows simply (see Exercise 3.20). This completes the proof of (i).

It then follows that the binary tree generated by \( m + W_\tau \) is equal to the binary tree generated by \( X \). Hence \( X \overset{d}{=} m + W_\tau \), thus proving (ii).

The proof of (iii) is an exercise (Exercise 3.20).

QED

Note that for (i,ii) we have not used that the embedding process is BM. It is sufficient that it is a continuous time martingale with continuous paths that are unbounded above and below. However, essentially such a martingale is a Gaussian process, so the result is not as general as it looks. We will finally state a simple version of the second Skorokhod embedding theorem.

**Theorem 3.4.9** Let \( \{X_n\}_n \) be a sequence of i.i.d. integrable random variables. Put \( S_n = \sum_{k=1}^n X_k \). Then there exists a non-decreasing sequence \( \{\tau_n\}_n \) of a.s. finite stopping times, such that for all \( n \)

i) \( (W_{\tau_1}, W_{\tau_2}, \ldots, W_{\tau_n}) \overset{d}{=} (S_1, S_2, \ldots, S_n) \);

ii) \( \tau_1, \tau_2 - \tau_1, \tau_3 - \tau_3, \ldots, \tau_n - \tau_{n-1} \) are i.i.d. random variables with mean \( E X_1^2 \).

**Proof.** see Exercise 3.21.

QED

This theorem can be e.g. used for an alternative proof of the Central Limit Theorem, and for deriving the distribution of \( \max_{1 \leq k \leq n} S_k \).
3.5 Exercises

Exercise 3.1 Consider the Ornstein Uhlenbeck process in example 3.1.2(B). Show that the defined process is a Markov process which converges in distribution to an $N(0, \sigma^2/2\alpha)$ distributed random variable. If $X_0 \overset{d}{=} N(0, \sigma^2/2\alpha)$, show that $X_t \overset{d}{=} N(0, \sigma^2/2\alpha)$ (in other words: the $N(0, \sigma^2/2\alpha)$ distribution is an invariant distribution for the Markov process). Show that $X_t$ is a Gaussian process with the given mean and covariance functions.

Exercise 3.2 Consider geometric BM in example 3.1.2(C). Show that it is a Markov process. Characterise the values of $\mu$ for which it is a super- a sub- or a mere martingale. Compute the transition function.

Exercise 3.3 Complete the proof of Lemma 3.1.4.

Exercise 3.4 Let $W$ be a BM. Show that the reflected Brownian motion defined by $X = |X_0 + W|$ is a Markov process with respect to its natural filtration and compute its transition function. (Hint: calculate the conditional probability $P_{\nu}(X_t \in B | F^X_s)$ by conditioning further on $F^W_s$).

Exercise 3.5 Let $X$ be a Markov process with state space $E$ and transition function $(P_t)_{t \geq 0}$. Show that for every bounded, measurable function $f$ on $E$ and for all $t \geq 0$, the process $(P_{t-s}f(X_s))_{s \in [0,t]}$ is a martingale.

Exercise 3.6 Prove that $\mu_{t_1,\ldots,t_n}$ defined in the proof of Corollary 3.2.2 are probability measures that form a consistent system. Hint: for showing that they are probability measures, look at the proof of the Fubini theorem.

Exercise 3.7 Work out the details of the proof of Lemma 3.2.3.

Exercise 3.8 Show for the Poisson process $X$ with initial distribution $\nu = \delta_x$ in Example 3.1.5, that $X$ is a Markov process w.r.t. the natural filtration, with the transition function specified in the example.

Exercise 3.9 Show Corollary 3.3.6 that canonical Brownian motion has the strong Markov property.

Exercise 3.10 Prove Corollary 3.3.7 and Lemma 3.3.8.

Exercise 3.11 Prove Lemma 3.3.12. See hint in the ‘proof’.

Exercise 3.12 Prove Lemma 3.3.9 by proving the following steps.

i) Is $S_x$ a stopping time or an optional time? Using the Markov property, show that for every $x \in E$

$$P_x\{S_x > t + s\} = P_x\{S_x > t\}P_x\{S_x > s\},$$

for all $s, t \geq 0$.

ii) Conclude that there exists an $a \in [0, \infty]$, possibly depending on $x$, such that

$$P_x(S_x > t) = e^{-at}.$$
iii) Determine $a$ for the Markov jump process (in terms of $\lambda$ and the stochastic matrix $P$) (cf. Example 3.2.1) and for the Poisson process. Hint: compute $E_x S_x$.

iv) Given that the process starts in state $x$, what is the probability that the new state is $y$ after time $\sigma_x$ for Markov jump process?

v) Finally, complete the proof of the lemma.

Exercise 3.13 Consider the situation of Exercise 3.12. Suppose that $x \in E$ is a holding point, i.e. a point for which $a \in (0, \infty)$.

i) Observe that $S_x < \infty$, $P_x$-a.s. and that $\{X_{S_x} = x, \sigma_x < \infty\} \subseteq \{S_x \circ \theta_{S_x} = 0, \sigma_x < \infty\}$.

ii) Using the strong Markov property, show that $P_x \{X_{S_x} = x, S_x < \infty\} = P_x \{X_{S_x} = x, S_x < \infty\} P_x \{S_x = 0\}$.

iii) Conclude that $P_x \{X_{S_x} = x, S_x < \infty\} = 0$, i.e. a canonical Markov process with right-continuous paths, satisfying the strong Markov property can only leave a holding point by a jump.

Exercise 3.14 Prove Lemma 3.3.14 (ii). Hint: suppose that $Z$ is an $F^X_t$-measurable random variable and $\sigma$ an $(F^X_t)_t$-optional time. Show that $Z \circ \theta_\sigma$ is $F^X_{(\sigma+t)_t}$-measurable. First show this for $Z$ an indicator, then use the appropriate monotone class argument.

Exercise 3.15 Prove Lemma 3.4.3 and Theorem 3.4.4.

Exercise 3.16 Show for Example 3.4.1 that $X$ is a Markov process, and show the validity of the assertions stated. Explain which condition of Theorem 3.3.4 fails in this example.

Exercise 3.17 Show that the maps $\phi$ and $\psi$ in the proof of Theorem 3.4.1 are Borel measurable.

Exercise 3.18 i) Derive the expression for the joint density of BM and its running maximum given in Corollary 3.4.2.

ii) Let $W$ be a standard BM and $S_t$ its running maximum. Show that for all $t \geq 0$ and $x > 0$

$$P\{S_t \geq x\} = P\{\tau_x \leq t\} = 2P\{W_t \geq x\} = P\{|W_t| \geq x\}.$$

Exercise 3.19 Prove Corollary 3.4.5.

Exercise 3.20 Consider the construction of the binary tree in §3.4.4. Construct the binary tree for a random variable $X$, that has a uniform distribution on $[0,1]$.

Consider Theorem 3.4.8. Show how the validity of statements (i,ii) follow for a general integrable random variable given that these statements are true for non-negative or non-positive integrable random variables. Show (iii).

Exercise 3.21 Prove Theorem 3.4.9.